On the Zeros of Polynomials and Bernstein Type Inequalities for Polynomials and Related Entire Functions

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A Dissertation

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Doctor of Philosophy

Auburn, Alabama

August 30, 1991

Vita

Robert Bentley Gardner, Junior, son of Robert Gardner and Ruby (Gordon) Gardner, was born February 4, 1963, in Montgomery, Alabama. He graduated from Hooper Academy in Hope Hull, Alabama in 1981. In June, 1981, he entered Auburn University at Montgomery and received the degree of Bachelor of Science (Mathematics) in May, 1984. While an undergraduate, he worked at the Alabama Department of Environmental Management, Air Division. He entered the graduate school of Auburn University in September, 1984. He did intermittent work in biology and physics and taught numerous mathematics classes. He received the degree Master of Science (Mathematics-Combinatorics) in August, 1987. He continued graduate work at Auburn University, concentrating on a Ph.D. in mathematics (Complex Analysis) and a Master of Science in biology (Population Genetics). He has accepted a position in the Department of Mathematics at Louisiana State University in Shreveport and begins work there in August of 1991. He is married to E. Standish Paul-Gardner. Their son, Quincey Paul Gardner, was born on May 2, 1991, during the preparation of this dissertation.

Dissertation Abstract

On the Zeros of Polynomials and Bernstein Type Inequalities for Polynomials and Related Entire Functions

Robert B. Gardner

Doctor of Philosophy, August 30, 1991
(M.S., Auburn University, 1987)
(B.S., Auburn University at Montgomery, 1984)

80 Typed Pages

Directed by Narendra K. Govil

In this dissertation, we explore four general topics. In the first chapter, we find regions containing the zeros of polynomials as functions of their coefficients. All results concern the moduli of the zeros, as opposed to, say, the arguments of the zeros. In the second chapter, we define a norm on the space of all polynomials $p(z)$ of degree less than or equal to n by $\|p\| = \max_{|z|=1} |p(z)|$ and then estimate the norm of the derivative of the polynomials in terms of the norm of the polynomial and its degree n. The third chapter contains results concerning the estimate of the maximum modulus of a polynomial on $|z| = r$ in terms of its maximum modulus on $|z| = 1$. Results are presented for both $r > 1$ and $r < 1$. Finally, in the fourth chapter, we present results for entire functions of exponential type which are generalizations of some of the results presented in Chapters 2 and 3 for polynomials. We also deal with a generalization of the differentiation operator.

Acknowledgments

The author would like to acknowledge "the gang" throughout the years. This includes Coker, Saad, Bollschweiler, P.R. and Eve, Marvin, Jeffey, Timmy, Becky, the Rickster, Marge, Ed, Earring, Evan, Momma G's, the Rabbit, John Denver, the state of Alaska, and the North Cascade Mountain Range. These guys made it a bit more than just bearable. A special thanks to Johnny for doing all that stuff the way it should be done. Also, his gratitude to Rosemary Kennington. She was "that one special teacher" (1977-1979).

He would like to give a special thanks to Dr. N. K. Govil for his time and patience through the years and also to the Department of Algebra, Combinatorics and Analysis for generous financial support even while his interests wandered from math.

Finally, he would like to thank my family, Ma, Pa, and Poo, for their financial, emotional, and logistic support.

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Chapter 1

The Location of the Zeros of a Polynomial

A classical problem has been to find an algebraic equation for the zeros of an n^{th} degree polynomial as a function of the coefficients of the polynomial. Unfortunately, as shown by the insolvability of the quintic, this in general cannot be done. So, a natural question to ask is "For a given polynomial $p(z) = \sum_{v=0}^{n} a_v z^v$, what restrictions can be put on the location (in the complex plane) of the zeros of $p(z)$?"

A well known result in this direction is due to Cauchy [10]:

Theorem 1.1 *All the zeros of* $p(z) = \sum_{v=0}^{n} a_v z^v$, *where* $a_n \neq 0$, *lie in the circle* $|z| < 1 + M$, *where* $M = \max_{0 \leq j \leq (n-1)} \left| \frac{a_j}{a_n} \right|$.

The following result which is an improvement of Theorem 1.1 is due to Kuniyeda [33]. The proof is based on Hölder's inequality (see Marden [40]):

Theorem 1.2 *For any* q *and* r *such that* $q > 1$, $r > 1$, *and* $\frac{1}{q} + \frac{1}{r} = 1$, *the polynomial* $p(z) = \sum_{v=0}^{n} a_v z^v$, *where* $a_n \neq 0$, *has all its zeros in the circle*

$$|z| < \left\{ 1 + \left[\sum_{v=0}^{n-1} \left| \frac{a_v}{a_n} \right|^q \right]^{r/q} \right\}^{1/r} \leq \left(1 + n^{r/q} M^r \right)^{1/r},$$

where $M = \max_{0 \leq j \leq (n-1)} \left| \frac{a_j}{a_n} \right|$.

This result was also proved independently by Montel [44], and Tôya [60].

Using Theorem 1.2, we can in fact obtain an annulus containing all the zeros of a polynomial. The following theorem is thus an improvement of Theorem 1.2.

Theorem 1.3 *For any* q *and* r *such that* $q > 1$, $r > 1$, *and* $\frac{1}{q} + \frac{1}{r} = 1$, *the polynomial* $p(z) = \sum_{v=0}^{n} a_v z^v$, *where* $a_0 \neq 0$, *has all its zeros in*

$$\left(1 + n^{q/r} M_1^r\right)^{-1/r} \leq \left\{1 + \left[\sum_{v=1}^{n} \left|\frac{a_v}{a_0}\right|^{r/q}\right]\right\}^{-1/r} < |z|$$
$$< \left\{1 + \left[\sum_{v=0}^{n-1} \left|\frac{a_v}{a_n}\right|^{q}\right]^{r/q}\right\}^{1/r} \leq \left(1 + n^{r/q} M_2^r\right)^{1/r}$$

where $M_1 = \max_{1 \leq j \leq n} \left|\frac{a_j}{a_0}\right|$ *and* $M_2 = \max_{0 \leq j \leq (n-1)} \left|\frac{a_j}{a_j}\right|$.

Proof. Consider the polynomial $P(z) = z^n p\left(\frac{1}{z}\right) = a_n + a_{n-1}z + \cdots + a_1 z^{n-1} + a_0 z^n$. By Theorem 1.2, $P(z)$ has all its zeros in

$$|z| < \left\{1 + \left[\sum_{v=1}^{n} \left|\frac{a_v}{a_0}\right|^{q}\right]^{r/q}\right\}^{1/r} \leq \left(1 + n^{r/q} M_1^r\right)^{1/r}$$

where M, q, and r are as described in this theorem. So $p\left(\frac{1}{z}\right)$ also has all its zeros in the same region. Replacing $\frac{1}{z}$ with z, we get that $p(z)$ has its zeros in

$$\left(1 + n^{q/r} M_1^r\right)^{-1/r} \leq \left\{1 + \left[\sum_{v=1}^{n} \left|\frac{a_v}{a_0}\right|^{r/q}\right]\right\}^{-1/r} < |z|$$

which when combined with Theorem 1.2 gives Theorem 1.3. □

Using an extension of Hölder's inequality, Jain [30] sharpened Theorem 1.2. He proved:

Theorem 1.4 *With the hypotheses of Theorem 1.2, $p(z)$ has all its zeros in*

$$|z| < R^{1/r},$$

where R is the unique root of the equation

$$x^3 - (1 + DN)x^2 + DNx - D = 0$$

in the interval $(1, \infty)$. Here

$$D = \left\{ \sum_{v=0}^{n-1} \left| \frac{a_v}{a_n} \right|^q \right\}^{r/q}$$

and

$$N = (|a_{n-1}| + |a_{n-2}|)^r \left(|a_{n-1}|^q + |a_{n-2}|^q \right)^{-(r-1)}.$$

Using the argument used in the proof of Theorem 1.3, one can easily obtain the following refinement of Theorem 1.4.

Theorem 1.5 *Under the hypotheses of Theorem 1.3, $p(z)$ has all its zeros in the annulus*

$$Q^{-1/r} < |z| < R^{1/r}$$

where R is the unique root of the equation

$$x^3 - (1 + D_1N_1)x^2 + D_1N_1x - D_1 = 0$$

in the interval $(1,\infty)$*, and* Q *is the unique root of the equation*

$$x^3-(1+D_2N_2)x^2+D_2N_2x-D_2=0$$

in the interval $(1,\infty)$*. Here*

$$D_1=\left\{\sum_{v=0}^{n-1}\left|\frac{a_v}{a_n}\right|^q\right\}^{r/q},$$

$$N_1=\left(|a_{n-1}|+|a_{n-2}|\right)^r\left(|a_{n-1}|^q+|a_{n-2}|^q\right)^{-(r-1)},$$

$$D_2=\left\{\sum_{v=1}^{n}\left|\frac{a_v}{a_0}\right|^q\right\}^{r/q}$$

and

$$N_2=\left(|a_1|+|a_2|\right)^r\left(|a_1|^q+|a_2|^q\right)^{-(r-1)}.$$

If we put some restrictions on the coefficients of $p(z)$, the above results can be improved. In particular, if all the coefficients are real and positive, the location of the zeros can again be restricted to an annulus. Kakeya [32], Hayashi [27], and Hurwitz [29] proved the following improvement of Theorem 1.1.

Theorem 1.6 *All the zeros of* $p(z)=\sum_{v=0}^{n}a_vz^v$*, where* a_j *are real and positive for* $j=0,1,\ldots,n$*, lie in*

$$R_1\le|z|\le R_2$$

where $R_1=\min_{0\le j\le(n-1)}\left(\frac{a_j}{a_{j+1}}\right)$ *and* $R_2=\max_{0\le j\le(n-1)}\left(\frac{a_j}{a_{j+1}}\right)$.

In particular, if the coefficients are nonnegative and monotonic increasing, then we have the following well known Eneström-Kakeya theorem [15, 32]:

Theorem 1.7 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *with real coefficients satisfying*

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_n,$$

then all the zeros of $p(z)$ *lie in* $|z| \leq 1$*.*

In 1967, Joyal, Labelle, and Rahman [31] dropped the condition that the coefficients be all nonnegative and proved the following.

Theorem 1.8 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *with real coefficients,* $a_n \neq 0$*, satisfying*

$$a_0 \leq a_1 \leq \cdots \leq a_n,$$

then all the zeros of $p(z)$ *lie in*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Notice that Theorem 1.8 reduces to Theorem 1.7 when $a_0 \geq 0$.

We can weaken the hypotheses of Theorem 1.8 and consider a larger class of polynomials. We are inspired by the work of Aziz and Mohammad [4]. In 1980, they presented the following two theorems for analytic functions, with an interesting and rather flexible condition on the coefficients of the series expansion of the function.

Theorem 1.9 *Let* $f(z) = \sum_{v=0}^{\infty} a_v z^v$ *be analytic in* $|z| \le t$*. If* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, \ldots$ *and for some* k*,*

$$0 < \alpha_0 \le t\alpha_1 \le \cdots \le t^k\alpha_k \ge t^{k+1}\alpha_{k+1} \ge \cdots$$

then $f(z) \ne 0$ *in*

$$|z| < t \bigg/ \left(\frac{2\alpha_k}{\alpha_0} t^k - 1 + \frac{2}{\alpha_0} \sum_{j=0}^{\infty} |\beta_j| t^j \right).$$

Theorem 1.10 *Let* $f(z) = \sum_{v=0}^{\infty} a_v z^v$ *be analytic in* $|z| \le t$*. If* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, \ldots$ *and for some* k *and* r*,*

$$0 < \alpha_0 \le t\alpha_1 \le \cdots \le t^k\alpha_k \ge t^{k+1}\alpha_{k+1} \ge \cdots$$

and

$$\beta_0 \le t\beta_1 \le \cdots \le t^r\beta_r \ge t^{r+1}\beta_{r+1} \ge \cdots$$

then $f(z) \ne 0$ *in*

$$|z| < \frac{t|a_0|}{\alpha_0 + \beta_0} \bigg/ \left(\frac{2(\alpha_k t^k + \beta_r t^r)}{\alpha_0 + \beta_0} - 1 \right).$$

Since our interest lies in polynomials, we will now put these types of restrictions on the coefficients of polynomials. We prove:

Theorem 1.11 *Suppose* $p(z) = \sum_{v=0}^{n} a_v z^v$, $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, \ldots, n$, $a_n \neq 0$ *and for some* k,

$$\alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq t^{k+2}\alpha_{k+2} \geq \cdots \geq t^n\alpha_n$$

for some positive t. *Then* $p(z)$ *has all its zeros in* $R_1 \leq |z| \leq R_2$ *where*

$$R_1 = \min\left\{ t|a_0| \Bigg/ \left(2t^k\alpha_k - \alpha_0 - t^n\alpha_n + |a_n|t^n + |\beta_0| + 2\sum_{j=1}^{n-1} |\beta_j|t^j + |\beta_n|t^n \right) \ , t \right\}$$

and

$$\begin{aligned} R_2 \quad = \quad & \max\left\{ \left(|a_0|t^{n+1} + (t^2+1)t^{n-k-1}\alpha_k - t^{n-1}\alpha_0 - t\alpha_n + (t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j \right.\right. \\ & \left.\left. +(1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{n} (|\beta_{j-1}| + t|\beta_j|)t^{n-j} \right) \Bigg/ |a_n| \, , \, \frac{1}{t} \right\}. \end{aligned}$$

Proof. Consider the polynomial

$$P(z) = (t-z)p(z) = ta_0 + \sum_{j=1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv -a_n z^{n+1} + G_2(z).$$

We first note that

$$|a_{j-1} - ta_j| = |\alpha_{j-1} - t\alpha_j + i(\beta_{j-1} - t\beta_j)| \leq |\alpha_{j-1} - t\alpha_j| + |\beta_{j-1}| + t|\beta_j|. \tag{1.1}$$

Then

$$\left| z^n G_2\left(\frac{1}{z}\right) \right| = \left| ta_0 z^n + \sum_{j=1}^{n} (ta_j - a_{j-1}) z^{n-j} \right|$$

and on $|z| = t$,

$$\begin{aligned}
\left|z^n G_2\left(\frac{1}{z}\right)\right| &\le |ta_0|t^n + \sum_{j=1}^{n} |ta_j - a_{j-1}|t^{n-j} \\
&\le |a_0|t^{n+1} + \sum_{j=1}^{n} |t\alpha_j - \alpha_{j-1}|t^{n-j} + \sum_{j=1}^{n} \left(|\beta_{j-1}| + t|\beta_j|\right) t^{n-j} \qquad \text{by (1.1)} \\
&= |a_0|t^{n+1} + \sum_{j=1}^{k} (t\alpha_j - \alpha_{j-1})t^{n-j} + \sum_{j=k+1}^{n} (\alpha_{j-1} - t\alpha_j)t^{n-j} \\
&\quad + \sum_{j=1}^{n} (|\beta_{j-1}| + t|\beta_j|)t^{n-j} \\
&= |a_0|t^{n+1} + (t^2+1)t^{n-k-1}\alpha_k - t^{n-1}\alpha_0 - t\alpha_n \\
&\quad + (t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + (1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j \\
&\quad + \sum_{j=1}^{n} (|\beta_{j-1}| + t|\beta_j|)t^{n-j} \\
&\equiv M_2. \qquad (1.2)
\end{aligned}$$

Hence, by the Maximum Modulus Principle (see, for example, p. 134 of Ahlfors [1]),

$$\left|z^n G_2\left(\frac{1}{z}\right)\right| \le M_2 \text{ for } |z| \le t$$

which implies

$$|G_2(z)| \le M_2|z|^n \text{ for } |z| \ge \frac{1}{t}.$$

From this follows

$$\begin{aligned} |P(z)| &= |-a_n z^{n+1} + G_2(z)| \\ &\geq |a_n||z|^{n+1} - M_2|z|^n \\ &= |z|^n(|a_n||z| - M_2) \text{ for } |z| \geq \frac{1}{t}. \end{aligned}$$

So if $|z| > \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\} \equiv R_2$, then $P(z) \neq 0$ and in turn $p(z) \neq 0$, thus establishing the outer radius for the theorem.

For the inner bound, consider

$$P(z) = (t - z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv ta_0 + G_1(z).$$

Then for $|z| = t$,

$$\begin{aligned} |G_1(z)| &\leq \sum_{j=1}^{n} |a_{j-1} - ta_j|t^j + |a_n|t^{n+1} \\ &\leq \sum_{j=1}^{n} |\alpha_{j-1} - t\alpha_j|t^j + \sum_{j=1}^{n}(|\beta_{j-1}| + t|\beta_j|)t^j + |a_n|t^{n+1} \qquad \text{by (1.1)} \\ &= -t\alpha_0 + 2t^{k+1}\alpha_k - t^{n+1}\alpha_n + |\beta_0|t + 2\sum_{j=1}^{n-1}|\beta_j|t^{j+1} + |\beta_n|t^{n+1} + |a_n|t^{n+1} \\ &\equiv M_1. \end{aligned}$$

Applying Schwarz's Lemma (see, for example, p. 168 of Titchmarsh [59]) to $G_1(z)$, we get

$$|G_1(z)| \leq \frac{M_1|z|}{t} \text{ for } |z| \leq t.$$

So

$$|P(z)| = |-ta_0 + G_1(z)| \geq t|a_0| - |G_1(z)| \geq t|a_0| - \frac{M_1|z|}{t}.$$

So if $|z| < \min\left\{\frac{t^2|a_0|}{M_1}, t\right\} \equiv R_1$ then $P(z) \neq 0$ and in turn $p(z) \neq 0$. □

If we let $\beta_j = 0$ for all j in Theorem 1.11, then we get the following extension of Theorem 1.8:

Corollary 1.1 *Suppose* $p(z) = \sum_{v=0}^{n} a_v z^v$, *is a polynomial of degree* n *with real coefficients and that for some* k *and some* $t > 0$,

$$a_0 \leq ta_1 \leq t^2 a_2 \leq \cdots \leq t^k a_k \geq t^{k+1} a_{k+1} \geq t^{k+2} a_{k+2} \geq \cdots \geq t^n a_n.$$

Then $p(z)$ *has all its zeros in* $R_1 \leq |z| \leq R_2$, *where*

$$R_1 = \min\left\{\frac{t|a_0|}{2t^k a_k - a_0 - t^n a_n + |a_n| t^n}\,, t\right\}$$

and

$$R_2 = \max\left\{\left(|a_0| t^{n+1} + (t^2+1) t^{n-k-1} a_k - t^{n-1} a_0 - t a_n \sum_{v=1}^{n} ** \right.\right.$$

$$+(t^2-1)\sum_{j=1}^{k-1} t^{n-j-1} a_j + (1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1} a_j \Bigg) \Bigg/ |a_n|\,,\ \frac{1}{t} \Bigg\}.$$

If we let $t = 1$ and $k = n$, then Corollary 1.1 implies Theorem 1.8. In fact, we get an annulus containing all the zeros. We get that all the zeros of $p(z)$ lie in

$$\frac{|a_0|}{|a_n| + a_n - a_0} \le |z| \le \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

By putting a restriction on the imaginary part of the coefficients, we can improve Theorem 1.11. The interest of the following theorem lies in its flexability. This is demonstrated in the four corollaries that follow from it.

Theorem 1.12 *Suppose* $p(z) = \sum_{v=0}^{n} a_v z^v$, $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, \ldots, n$, $a_n \neq 0$ *and for some* k *and* r *and for some* $t \geq 0$*, we have*

$$\alpha_0 \le t\alpha_1 \le t^2\alpha_2 \le \cdots \le t^k\alpha_k \ge t^{k+1}\alpha_{k+1} \ge t^{k+2}\alpha_{k+2} \ge \cdots \ge t^n\alpha_n$$

and

$$\beta_0 \le t\beta_1 \le t^2\beta_2 \le \cdots \le t^r\beta_r \ge t^{r+1}\beta_{r+1} \ge t^{r+2}\beta_{r+2} \ge \cdots \ge t^n\beta_n.$$

Then $p(z)$ *has all its zeros in* $R_1 \le |z| \le R_2$ *where*

$$R_1 = \min\Big\{ t|a_0| \Big/ \Big(-(\alpha_0+\beta_0) + 2(t^k\alpha_k + t^r\beta_r) - t^n(\alpha_n + \beta_n - |a_n|) \Big)\ ,t \Big\}$$

and

$$
\begin{aligned}
R_2 &= \max\Bigg\{\Bigg(\Bigg(|a_0|t^{n+1} - t^{n-1}(\alpha_0+\beta_0) - t(\alpha_n+\beta_n) + (t^2+1)(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r)\sum_{v=1}^{n} ** \\
&\quad +(t^2-1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right) \\
&\quad +(1-t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right)\Bigg)\Bigg) \Big/ |a_n|\,,\ \frac{1}{t}\Bigg\}.
\end{aligned}
$$

Proof. Consider the polynomial

$$
P(z) = (t-z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv -a_n z^{n+1} + G_2(z).
$$

Then

$$
\left|z^n G_2\left(\frac{1}{z}\right)\right| = \left|ta_0 z^n + \sum_{j=1}^{n}(ta_j - a_{j-1})z^{n-j}\right|
$$

and on $|z| = t$,

$$
\begin{aligned}
\left|z^n G_2\left(\frac{1}{z}\right)\right| &\le |ta_0|t^n + \sum_{j=1}^{n}|ta_j - a_{j-1}|t^{n-j} \\
&\le |a_0|t^{n+1} + \sum_{j=1}^{n}|t\alpha_j - \alpha_{j-1}|t^{n-j} + \sum_{j=1}^{n}\left(|t\beta_j - t\beta_{j-1}|\right)t^{n-j} \\
&= |a_0|t^{n+1} + \sum_{j=1}^{k}(t\alpha_j - \alpha_{j-1})t^{n-j} + \sum_{j=k+1}^{n}(\alpha_{j-1} - t\alpha_j)t^{n-j} \\
&\quad + \sum_{j=1}^{r}(t\beta_j - \beta_{j-1})t^{n-j} + \sum_{j=r+1}^{n}(\beta_{j-1} - t\beta_j)t^{n-j} \\
&= |a_0|t^{n+1} - t^{n-1}(\alpha_0+\beta_0) - t(\alpha_n+\beta_n) + (t^2+1)(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r) \\
&\quad +(t^2-1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right)
\end{aligned}
$$

$$\begin{aligned} & +(1-t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right) \\ \equiv \quad & M_2. \end{aligned} \tag{1.3}$$

Hence, by the Maximum Modulus Theorem,

$$\left|z^n G_2\left(\frac{1}{z}\right)\right| \leq M_2 \text{ for } |z| \leq t.$$

Which implies $|G_2(z)| \leq M_2|z|^n$ for $|z| \geq \frac{1}{t}$. From this follows

$$\begin{aligned} |P(z)| &= |-a_n z^{n+1} + G_2(z)| \\ &\geq |a_n||z|^{n+1} - M_2|z|^n \\ &= |z|^n(|a_n||z| - M_2) \text{ for } |z| \geq \frac{1}{t}. \end{aligned}$$

So if $|z| > \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\} \equiv R_2$, then $P(z) \neq 0$ and in turn $p(z) \neq 0$, again establishing the outer radius for the theorem.

For the inner bound, consider

$$P(z) = (t-z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv ta_0 + G_1(z).$$

On $|z| = t$,

$$|G_1(z)| \quad \leq \quad \sum_{j=1}^{k}(t\alpha_j - \alpha_{j-1})t^j + \sum_{j=k+1}^{n}(\alpha_{j-1} - t\alpha_j)t^j$$

$$
\begin{aligned}
& +\sum_{j=1}^{r}(t\beta_j-\beta_{j-1})t^j+\sum_{j=r+1}^{n}(\beta_{j-1}-t\beta_j)t^j+|a_n|t^{n+1} \\
= \quad & -t(\alpha_0-\beta_0)+2(t^{k+1}\alpha_k+t^{r+1}\beta_r)-t^{n+1}(\alpha_n+\beta_n-|a_n|) \\
\equiv \quad & M_1.
\end{aligned}
$$

Applying Schwarz's Lemma to $G_1(z)$, we get

$$|G_1(z)| \leq \frac{M_1|z|}{t} \text{ for } |z| \leq t.$$

Which implies

$$|P(z)| = |-ta_0+G_1(z)| \geq t|a_0|-|G_1(z)| \geq t|a_0|-\frac{M_1|z|}{t}.$$

Hence if $|z| < \min\left\{\frac{t^2|a_0|}{M_1}, t\right\} \equiv R_1$ then $P(z) \neq 0$ and in turn $p(z) \neq 0$. □

Notice that if in the above theorem we take $t=1$, $\beta_j=0$ for all j, and $n=k$, we get Theorem 1.8. By making certain choices of t, k and r we obtain the following corollaries. In each, $p(z)=\sum_{v=0}^{n} a_v z^v$, $a_n \neq 0$, $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$ for $j=0,1,\ldots,n$. If in Theorem 1.12, we take $t=1$, $k=n$ and $r=n$, then we get:

Corollary 1.2 *If $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$ then $p(z)$ has all its zeros in*

$$\frac{|a_0|}{|a_n|-(\alpha_0+\beta_0)+(\alpha_n+\beta_n)} \leq |z| \leq \frac{|a_0|-(\alpha_0+\beta_0)+(\alpha_n+\beta_n)}{|a_n|}.$$

If in Theorem 1.12, we take $t = 1$, $k = 0$ and $r = 0$ then we get:

Corollary 1.3 *If $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ then $p(z)$ has all its zeros in*

$$\frac{|a_0|}{|a_n| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)}{|a_n|}.$$

If in Theorem 1.12, we take $t = 1$, $k = n$ and $r = 0$ then we get:

Corollary 1.4 *If $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$ then $p(z)$ has all its zeros in*

$$\frac{|a_0|}{|a_n| + \alpha_0 - \beta_0 - \alpha_n + \beta_n} \leq |z| \leq \frac{|a_0| + \alpha_0 - \beta_0 - \alpha_n + \beta_n}{|a_n|}.$$

Lastly, if in Theorem 1.12, we take $t = 1$, $k = 0$ and $r = n$ then we get:

Corollary 1.5 *If $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ then $p(z)$ has all its zeros in*

$$\frac{|a_0|}{|a_n| - \alpha_0 + \beta_0 + \alpha_n - \beta_n} \leq |z| \leq \frac{|a_0| - \alpha_0 + \beta_0 + \alpha_n - \beta_n}{|a_n|}.$$

The following result is a well known generalization of Schwarz's Lemma (see, for example, p. 167 of [45]).:

Lemma 1.1 *If $f(z)$ is analytic on $|z| \leq 1$, $|f(z)| \leq M$ on $|z| = 1$ and $f(0) = a$, then for $|z| \leq 1$*

$$|f(z)| \leq M \frac{M|z| + |a|}{|a||z| + M}.$$

The following result which is related to Lemma 1.1 is due to Govil, Rahman, and Schmeisser [26]:

Lemma 1.2 *If $f(z)$ is analytic in $|z| \leq R$, $f(0) = 0$, $|f'(0)| = b$ and $|f(z)| \leq M$ for $|z| = R$, then for $|z| \leq R$,*

$$|f(z)| \leq \frac{M|z|}{R^2} \frac{M|z| + R^2 b}{M + |z|b}.$$

By using the above lemmas, Dewan and Govil [12] were able to improve Theorem 1.8. They proved:

Theorem 1.13 *If $p(z) = \sum_{v=0}^{n} a_v z^v$, $a_n \neq 0$, $a_j \in \mathbf{R}$ for $j = 0, 1, \ldots, n$, and*

$$a_0 \leq a_1 \leq \cdots \leq a_n$$

then $p(z)$ has all its zeros in $R_1 \leq |z| \leq R_2$ where

$$R_2 = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_2}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_2}\right)^2 + \frac{M_2}{|a_n|}\right\}^{1/2}$$

and

$$R_1 = \frac{1}{2M_1^2}\left[-R_2^2 b(M_1 - |a_0|) + \left\{R_2^4 b^2 (M_1 - |a_0|)^2 + 4|a_0| R_2^2 M_1^3\right\}^{1/2}\right]$$

where

$$\begin{aligned} M_2 &= a_n - a_0 + |a_0|, \\ M_1 &= R_2^n(|a_n| R_2 + a_n - a_0), \\ c &= a_n - a_{n-1}, \quad \textit{and} \end{aligned}$$

$$b \;=\; a_1 - a_0.$$

Note that Theorem 1.13 is an improvement of Theorem 1.8 since as is shown in Dewan and Govil [12] that

$$R_2 \le \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Theorem 1.13 was extended to polynomials with complex coefficients. Dewan and Govil [13] proved the following:

Theorem 1.14 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$, $a_n \neq 0$, $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$, *for* $j = 0, 1, \ldots, n$, *and* $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n$ *then* $p(z)$ *has all its zeros in* $R_1 \le |z| \le R_2$ *where* R_1 *and* R_2 *are as given in Theorem 1.13 and*

$$\begin{aligned}
M_2 &= \alpha_n R_0 - (\alpha_0 + |\beta_0|) + |\alpha_0|, \\
M_1 &= R_2^n \left[\alpha_n R_0 + (|\alpha_n| + |\beta_n|)R_2 - (\alpha_0 + |\beta_0|)\right], \\
R_0 &= 1 + \frac{1}{\alpha_n}\left(2\sum_{j=0}^{n-1} |\beta_j| + |\beta_n|\right), \\
b &= |a_1 - a_0|, \quad \textit{and} \\
c &= |a_n - a_{n-1}|.
\end{aligned}$$

Note that Theorem 1.14 reduces to Theorem 1.13 when $\beta_j = 0$ for all j.

If we use Lemmas 1.1 and 1.2 instead of Schwarz's Lemma in the proof of Theorem 1.11, we get the following which we believe to be an improvement of the Theorem 1.11.

Theorem 1.15 *Under the hypotheses of Theorem 1.11, all the zeros of $p(z)$ lie in $R_1 \leq |z| \leq R_2$ where*

$$R_2 = \max\left\{\frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_2 t}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_2 t}\right)^2 + \frac{M_2}{|a_n| t}\right\}^{1/2}, \frac{1}{t}\right\}$$

and

$$R_1 = \min\left\{\frac{1}{2M_1^2}\left[-R_2^2 b(M_1 - |a_0|t) + \left\{R_2^4 b^2 (M_1 - |a_0|t)^2 + 4|a_0|tR_2^2 M_1^3\right\}^{1/2}\right], t\right\}$$

where

$$\begin{aligned}
M_2 &= |a_0|t^{n+1} + (t^2+1)t^{n-k-1}\alpha_k - t^{n-1}\alpha_0 - t\alpha_n + (t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j \\
&\quad + (1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{n}\left(|\beta_{j-1}| + t|\beta_j|\right)t^{n-j}, \\
M_1 &= R_2^k\left(t\alpha_k - \alpha_0 + (t-1)\sum_{j=1}^{k-1}\alpha_j\right) + R_2^n\left(\alpha_k - t\alpha_n + (1-t)\sum_{j=k+1}^{n-1}\alpha_j\right) \\
&\quad + \left(|\alpha_n| + |\beta_n|\right)R_2^{n+1} + R_2^n\sum_{j=1}^{n}\left(|\beta_{j-1}| + t|\beta_j|\right), \\
b &= |ta_1 - a_0|, \quad \textit{and} \\
c &= |ta_n - a_{n-1}|.
\end{aligned}$$

Proof. Consider the polynomial

$$P(z) = (t-z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv -a_n z^{n+1} + G_2(z).$$

From equation (1.2) we have $\left|z^n G_2\left(\frac{1}{z}\right)\right| \leq M_2$ for $|z| = t$. Also, $z^n G_2\left(\frac{1}{z}\right)$ yields $ta_n - a_{n-1}$ when evaluated at $z=0$. So by Lemma 1.1 for $|z| \leq t$,

$$\left|z^n G_2\left(\frac{1}{z}\right)\right| \leq M_2 \frac{M_2\frac{|z|}{t} + c}{c\frac{|z|}{t} + M_2} = M_2 \frac{M_2|z| + tc}{c|z| + tM_2},$$

where $c = |ta_n - a_{n-1}|$. Which gives for $|z| \geq \frac{1}{t}$,

$$|G_2(z)| \leq M_2 |z|^n \frac{M_2 + ct|z|}{c + M_2 t|z|}.$$

Therefore,

$$\begin{aligned} |P(z)| &= |-a_n z^{n+1} + G_2(z)| \geq |a_n||z|^{n+1} - |G_2(z)| \\ &\geq |a_n||z|^{n+1} - M_2|z|^n \frac{M_2 + ct|z|}{c + M_2 t|z|} \\ &= \frac{|z|^n}{M_2 t|z| + c}\left\{tM_2|a_n||z|^2 - c(M_2 t - |a_n|)|z| - M_2^2\right\}. \end{aligned}$$

And so $|P(z)| > 0$, and, in turn $|p(z)| > 0$, if

$$\begin{aligned} |z| &> \max\left\{\frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_2 t}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_2 t}\right)^2 + \frac{M_2}{|a_n|t}\right\}^{1/2}, \frac{1}{t}\right\} \\ &\equiv R_2. \end{aligned} \tag{1.4}$$

For the inner bound, consider

$$P(z) = (t-z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv ta_0 + G_1(z).$$

Then

$$\begin{aligned}
|G_1(z)| &= \left|\sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1}\right| \\
&\leq \sum_{j=1}^{n}|t\alpha_j - \alpha_{j-1}||z|^j + \sum_{j=1}^{n}(|\beta_{j-1}| + t|\beta_j|)|z|^j + |a_n||z|^{n+1} \qquad \text{by (1.1)} \\
&= \sum_{j=1}^{k}(t\alpha_j - \alpha_{j-1})|z|^j + \sum_{j=k+1}^{n}(\alpha_{j-1} - t\alpha_j)|z|^j \\
&\quad + \sum_{j=1}^{n}(|\beta_{j-1} + t|\beta_j|)|z|^j + |a_n||z|^{n+1}.
\end{aligned}$$

On $|z| = R_2 \geq 1$,

$$\begin{aligned}
|G_1(z)| &\leq R_2^k \sum_{j=1}^{k}(t\alpha_j - \alpha_{j-1}) \\
&\quad + R_2^n \sum_{j=k+1}^{n}(\alpha_{j-1} - t\alpha_j) + R_2^n \sum_{j=1}^{n}(|\beta_{j-1}| + t|\beta_j|) + |a_n|R_2^{n+1} \\
&= R_2^k\left(t\alpha_k - \alpha_0 + (t-1)\sum_{j+1}^{k-1}\alpha_j\right) + R_2^n\left(\alpha_k - t\alpha_n + (1-t)\sum_{j=k+1}^{n-1}\alpha_j\right) \\
&\quad + R_2^n \sum_{j=1}^{n}(|\beta_{j-1}| + t|\beta_j|) + |a_n|R_2^{n+1} \\
&\leq R_2^k\left(t\alpha_k - \alpha_0 + (t-1)\sum_{j+1}^{k-1}\alpha_j\right) + R_2^n\left(\alpha_k - t\alpha_n + (1-t)\sum_{j=k+1}^{n-1}\alpha_j\right) \\
&\quad + R_2^n \sum_{j=1}^{n}(|\beta_{j-1}| + t|\beta_j|) + (|\alpha_n| + |\beta_n|)R_2^{n+1}
\end{aligned}$$

$$\equiv \quad M_1.$$

Now, $G_1(0) = 0$ and $|G_1'(0)| = |ta_1 - a_0| \equiv b$ and so by Lemma 1.2,

$$|G_1(z)| \leq \frac{M_1|z|}{R_2^2}\frac{M_1|z| + R_2^2 b}{M_1 + |z|b} \text{ for } |z| \leq R_2.$$

Hence for $|z| \leq R_2$, we have

$$\begin{aligned}
|P(z)| &= |ta_0 + G_1(z)| \geq |ta_0| - |G_1(z)| \\
&\geq |ta_0| - \frac{M_1|z|}{R_2^2}\frac{M_1|z| + R_2^2 b}{M_1 + |z|b} \\
&= \frac{-1}{R_2^2(M_1 + |z|b)}\left\{M_1^2|z|^2 + R_2^2 b(M_1 - t|a_0|)|z| - t|a_0|R_2^2 M_1\right\},
\end{aligned}$$

which implies that $|P(z)| > 0$ and in turn $|p(z)| > 0$ if

$$\begin{aligned}
|z| &< \min\left\{\frac{1}{2M_1^2}\left[-R_2^2 b(M_1 - |a_0|t) + \left\{R_2^4 b^2(M_1 - |a_0|t)^2 + 4|a_0|tR_2^2 M_1^3\right\}^{1/2}\right], t\right\} \\
&\equiv R_1. \qquad (1.5)
\end{aligned}$$

This establishes the theorem. □

Note that Theorem 1.15 reduces to Theorem 1.14 when $t = 1$ and $n = k$. If we let $\beta_j = 0$ for all j in Theorem 1.15, we get the following which sharpens Corollary 1.1 and generalizes Theorem 1.13.

Corollary 1.6 *Under the hypotheses of Corollary 1.1, all the zeros of* $p(z)$ *lie in* $R_1 \leq |z| \leq R_2$ *where* R_1 *and* R_2 *are as given in Theorem 1.13 and*

$$\begin{aligned} M_2 &= |a_0|t^{n+1} + (t^2+1)t^{n-k-1}a_k - t^{n-1}a_0 - ta_n + (t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}a_j \\ &\quad +(1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}a_j, \\ M_1 &= R_2^k\left(ta_k - a_0 + (t-1)\sum_{j=1}^{k-1} a_j\right) + R_2^n\left(a_k - ta_n + (1-t)\sum_{j=k+1}^{n-1} a_j\right) + |a_n|R_2^{n+1}, \\ b &= |ta_1 - a_0|, \quad \textit{and} \\ c &= |ta_n - a_{n-1}|. \end{aligned}$$

If we let $t = 1$ and $k = n$, Corollary 1.6 reduces to Theorem 1.13.

Our next result which we believe is a refinement of Theorem 1.12 is obtained by using Lemmas 1.1 and 1.2.

Theorem 1.16 *Under the hypotheses of Theorem 1.12, all the zeroes of* $p(z)$ *lie in* $R_1 \leq |z| \leq R_2$ *where* R_1 *and* R_2 *are as given in Theorem 1.15 and*

$$\begin{aligned} M_2 &= |a_0|t^{n+1} - t^{n-1}(\alpha_0+\beta_0) - t(\alpha_n+\beta_n) + (t^2+1)(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r) \\ &\quad +(t^2-1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right) \\ &\quad +(1-t^2)\left(\sum_{j=k+1}^{n} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n} t^{n-j-1}\beta_j\right), \\ M_1 &= R_2^k\left(t\alpha_k - \alpha_0 + (t-1)\sum_{j=1}^{k-1}\alpha_j\right) + R_2^r\left(t\beta_r - \beta_0 + (t-1)\sum_{j=1}^{r-1}\beta_j\right) \end{aligned}$$

$$
\begin{aligned}
& +R_2^n\left(\alpha_k - t\alpha_n + \beta_r - t\beta_n + (1-t)\sum_{j=k+1}^{n-1}\alpha_j + (1-t)\sum_{j=r+1}^{n-1}\beta_j\right) + |a_n|R_2^{n+1}, \\
b &= |ta_1 - a_0|, \quad \textit{and} \\
c &= |ta_n - a_{n-1}|.
\end{aligned}
$$

Proof. Consider the polynomial

$$P(z) = (t-z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv -a_n z^{n+1} + G_2(z).$$

From equation (1.3) we have $\left|z^n G_2\left(\frac{1}{z}\right)\right| \leq M_2$ for $|z| = t$. Also, $z^n G_2\left(\frac{1}{z}\right)$ yields $ta_n - a_{n-1}$ when evaluated at $z = 0$. So we get from Lemma 1.1 that for $|z| \leq t$,

$$\left|z^n G_2\left(\frac{1}{z}\right)\right| \leq M_2 \frac{M_2\frac{|z|}{t} + c}{c\frac{|z|}{t} + M_2} = M_2 \frac{M_2|z| + tc}{c|z| + tM_2},$$

where $c = |ta_n - a_{n-1}|$. Hence,

$$|G_2(z)| \leq M_2|z|^n \frac{M_2 + ct|z|}{c + M_2 t|z|} \text{ for } |z| \geq \frac{1}{t}.$$

Therefore,

$$
\begin{aligned}
|P(z)| &= |-a_n z^{n+1} + G_2(z)| \geq |a_n||z|^{n+1} - |G_2(z)| \\
&\geq |a_n||z|^{n+1} - M_2|z|^n \frac{M_2 + ct|z|}{c + M_2 t|z|} \\
&= \frac{|z|^n}{M_2 t|z| + c}\left\{tM_2|a_n||z|^2 - c(M_2 t - |a_n|)|z| - M_2^2\right\}.
\end{aligned}
$$

From (1.4), $p(z) \neq 0$ if $|z| > R_2$.

For the inner bound, consider

$$P(z) = (t-z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1}z^j) - a_n z^{n+1} \equiv ta_0 + G_1(z).$$

Then

$$\begin{aligned}
|G_1(z)| &= \left|\sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1}\right| \\
&\leq \sum_{j=1}^{n}|t\alpha_j - \alpha_{j-1}||z|^j + \sum_{j=1}^{n}(t\beta_j - \beta_{j-1})|z|^j + |a_n||z|^{n+1} \\
&= \sum_{j=1}^{k}(t\alpha_j - \alpha_{j-1})|z|^j + \sum_{j=k+1}^{n}(\alpha_{j-1} - t\alpha_j)|z|^j \\
&\quad + \sum_{j=1}^{r}(t\beta_j - \beta_{j-1})|z|^j + \sum_{j=r+1}^{n}(\beta_{j-1} - t\beta_j)|z|^j + |a_n||z|^{n+1}.
\end{aligned}$$

And so for $|z| = R_2 \geq 1$,

$$\begin{aligned}
|G_1(z)| &\leq R_2^k\sum_{j=1}^{k}(t\alpha_k - \alpha_{j-1}) + R_2^n\sum_{j=k+1}^{n}(\alpha_{j-1} - t\alpha_j) \\
&\quad + R_2^r\sum_{j=1}^{r}(t\beta_j - \beta_{j-1}) + R_2^n\sum_{j=r+1}^{n}(t\beta_j - \beta_{j-1}) + |a_n|R_2^{n+1} \\
&= R_2^k\left(t\alpha_k - \alpha_0 + (t-1)\sum_{j=1}^{k-1}\alpha_j\right) + R_2^r\left(t\beta_r - \beta_0 + (t-1)\sum_{j=1}^{r-1}\beta_j\right) \\
&\quad + R_2^n\left(\alpha_k - t\alpha_n + \beta_r - t\beta_n + (1-t)\sum_{j=k+1}^{n-1}\alpha_j + (1-t)\sum_{j=r+1}^{n-1}\beta_j\right)
\end{aligned}$$

$$
\begin{aligned}
& +|a_n|R_2^{n+1} \\
\equiv \quad & M_1.
\end{aligned}
$$

Now, $G_1(0) = 0$ and $|G_1'(0)| = |ta_1 - a_0| \equiv b$ and so by Lemma 1.2,

$$|G_1(z)| \leq \frac{M_1|z|}{R_2^2}\frac{M_1|z| + R_2^2 b}{M_1 + |z|b}$$

for $|z| \leq R_2$. By equation (1.5), $p(z) \neq 0$ if

$$
\begin{aligned}
|z| \quad < \quad & \frac{1}{2M_1^2}\left[-R_2^2 b(M_1 - |a_0|t) + \left\{R_2^4 b^2(M_1 - |a_0|t)^2 + 4|a_0|tR_2^2 M_1^3\right\}^{1/2}\right] \\
\equiv \quad & R_1.
\end{aligned}
$$

This establishes the theorem. □

If in Theorem 1.16 we take $\beta_j = 0$ for all j, it reduces to Corollary 1.6. By taking certain choices of t, k and r, we obtain the following corollaries. In each, $p(z) = \sum_{v=0}^{n} a_v z^v$, $a_n \neq 0$, and $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$ for $j = 0, 1, \ldots, n$. We believe that these results sharpen Corollaries 1.2 through 1.5, respectively. If in Theorem 1.16, we take $t = 1$, $k = n$ and $r = n$ then we get:

Corollary 1.7 *If $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n$ and $\beta_0 \le \beta_1 \le \cdots \le \beta_n$ then $p(z)$ has all its zeros in $R_1 \le |z| \le R_2$ where R_1 and R_2 are as given in Theorem 1.15 and*

$$\begin{aligned} M_2 &= |a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n), \\ M_1 &= R_2^n \left[-(\alpha_0 + \beta_0) + (\alpha_n + \beta_n) \right] + |a_n| R_2^{n+1}, \\ b &= |ta_1 - a_0|, \quad \textit{and} \\ c &= |ta_n - a_{n-1}|. \end{aligned}$$

If in Theorem 1.16, we take $t = 1$, $k = 0$ and $r = 0$ then we get:

Corollary 1.8 *If $\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_n$ and $\beta_0 \ge \beta_1 \ge \cdots \ge \beta_n$ then $p(z)$ has all its zeros in $R_1 \le |z| \le R_2$ where R_1 and R_2 are as given in Theorem 1.15 and*

$$\begin{aligned} M_2 &= |a_0| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n), \\ M_1 &= R_2^n \left[(\alpha_0 + \beta_0) - (\alpha_n + \beta_n) \right] + |a_n| R_2^{n+1}, \\ b &= |ta_1 - a_0|, \quad \textit{and} \\ c &= |ta_n - a_{n-1}|. \end{aligned}$$

If in Theorem 1.16, we take $t = 1$, $k = n$ and $r = 0$ then we get:

Corollary 1.9 *If* $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$ *and* $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$ *then* $p(z)$ *has all its zeros in* $R_1 \leq |z| \leq R_2$ *where* R_1 *and* R_2 *are as given in Theorem 1.15 and*

$$\begin{aligned} M_2 &= |a_0| - \alpha_0 + \beta_0 + \alpha_n - \beta_n, \\ M_1 &= R_2^n(\alpha_0 - \beta_0 - \alpha_n + \beta_n) + |a_n| R_2^{n+1}, \\ b &= |ta_1 - a_0|, \quad \textit{and} \\ c &= |ta_n - a_{n-1}|. \end{aligned}$$

Lastly, if in Theorem 1.16, we take $t = 1$, $k = 0$ and $r = n$ then we get:

Corollary 1.10 *If* $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ *and* $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ *then* $p(z)$ *has all its zeros in* $R_1 \leq |z| \leq R_2$ *where* R_1 *and* R_2 *are as given in Theorem 1.15 and*

$$\begin{aligned} M_2 &= |a_0| + \alpha_0 - \beta_0 - \alpha_n + \beta_n, \\ M_1 &= R_2^n(-\alpha_0 + \beta_0 + \alpha_n - \beta_n) + |a_n| R_2^{n+1}, \\ b &= |ta_1 - a_0|, \quad \textit{and} \\ c &= |ta_n - a_{n-1}|. \end{aligned}$$

Chapter 2

The Norm of the Derivative of a Polynomial in terms of the Norm of the Polynomial

In 1887, the Russian chemist Dmitri Mendeleev studied the specific gravity of a solution as a function of the percentage of the dissolved substance [43]. He noticed that his data could be closely approximated by quadratic arcs and wondered if the corners where the arcs joined were actual, or were due to errors of measurement. If the slope of one arc exceeds the largest possible slope of an adjacent arc, then the arcs must come from different quadratic functions. After normalizations, the question becomes "If $p(x)$ is a quadratic polynomial with real coefficients and $|p(x)| \leq 1$ on $[-1, 1]$, then how large can $|p'(x)|$ be on $[-1, 1]$?" Mendeleev found that $|p'(x)| \leq 4$. This answer is best possible (or "sharp") as is shown by the example $p(x) = 1 - 2x^2$. With this result, Mendeleev decided that the corners in his data were genuine. For a complete historical review, see Boas [9].

Mendeleev corresponded with A. A. Markov about this result. Markov proved the following theorem which is the extension of Mendeleev's result to polynomials of arbitrary degree n [41].

Theorem 2.1 *If $p(x)$ is a polynomial of degree n with real coefficients, and $|p(x)| \leq M$ on $[-1, 1]$ then $|p'(x)| \leq Mn^2$ on $[-1, 1]$.*

The example $p(x) = \pm T_n(x) = \pm \cos(n \cos^{-1}(x))$, the n^{th} Chebyshev polynomial, shows that this result is sharp.

We now turn our attention to results for polynomials over the complex field. We also introduce a bit of notation:

Definition 2.1 *Let $\mathcal{P}_n$ denote the collection of all polynomials (over the complex field) of degree less than or equal to n. For $p \in \mathcal{P}_n$, let $\|p\| = \max_{|z|=1} |p(z)|$.*

The first result is due to Serge Bernstein [6]. When approximating functions with polynomials, he wanted the analogue of Theorem 2.1 for complex polynomials. He proved:

Theorem 2.2 *If $p \in \mathcal{P}_n$ then $\|p'\| \leq n\|p\|$.*

The result is sharp if and only if $p(z) = \alpha z^n$. So if we put some restrictions on the location of the zeros of $p(z)$, the bound of Theorem 2.2 can be improved. Paul Erdös conjectured and Peter Lax proved [35]:

Theorem 2.3 *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ in $|z| < 1$, then*

$$\|p'\| \leq \frac{n}{2}\|p\|.$$

This is also a sharp result, as is shown by the example $p(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

Theorem 2.3 also holds for *self inversive* polynomials, which satisfy the property $p(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ [5, 17, 46, 57]. In 1969, Malik [39] weakened the condition $p(z) \neq 0$ in $|z| < 1$ to prove:

Theorem 2.4 *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ for $|z| < K$, where $K \geq 1$, then*

$$\|p'\| \leq \frac{n}{1+K}\|p\|.$$

The bound is sharp and is obtained for $p(z) = \left(\frac{z+K}{1+K}\right)^n$. If $K = 1$ in Theorem 2.4, then Theorem 2.3 is obtained as a corollary. This result was generalized for the s^{th} derivative by Govil and Rahman [25]. For an inequality analagous to Theorem 2.2 for polynomials satisfying $p(z) = z^n p\left(\frac{1}{z}\right)$ see Govil, Jain and Labelle [23].

Another generalization of Theorem 2.3 was given by DeBruijn in 1947 (originally [11], see also [51]).

Definition 2.2 *For $p \in \mathcal{P}_n$, define the L_δ norm of p as*

$$\|p\|_\delta = \left\{\int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta\right\}^{1/\delta}.$$

DeBruijn's theorem is:

Theorem 2.5 *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ in $|z| < 1$ then for $\delta > 1$,*

$$\|p'\|_\delta \leq nC_\delta\|p\|_\delta$$

where

$$C_\delta = \left\{\frac{1}{2\pi}\int_0^{2\pi} |1+e^{i\alpha}|^\delta d\alpha\right\}^{-1/\delta} = 2^{-\delta}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}\delta+1)}{\Gamma(\frac{1}{2}\delta+\frac{1}{2})}.$$

If we let $\delta \to \infty$ in Theorem 2.5, we obtain Theorem 2.3 as a corollary. The proof of Theorem 2.5 depends on the following lemma, which is also due to DeBruijn [11]:

Lemma 2.1 *Let R be a convex region in the z-plane and let B be its boundary. Let $p(z)$ and $q(z)$ be polynomials with the roots of $q(z)$ in $R \bigcup B$ and let the degree of $p(z)$ be less than or equal to the degree of $q(z)$. If $|p(z)| \leq |q(z)|$ for $z \in B$, then $|p'(z)| \leq |q'(z)|$ for $z \in B$.*

Again, if we have more information about the location of the zeros of $p(z)$, the bound on $\|p'\|$ can be further refined. Govil and Labelle [24] used the information about the location of each of the zeros to prove:

Theorem 2.6 *If $p(z) = a_n \Pi_{v=1}^{n}(z - z_v)$ and $|z_v| \geq K_v \geq 1$ for $v = 1, 2, \ldots, n$, then*

$$\|p'\| \leq n \left(\frac{\sum_{v=1}^{n} \frac{1}{K_v - 1}}{\sum_{v=1}^{n} \frac{K_v+1}{K_v-1}} \right) \|p\| = \frac{n}{2} \left(1 - \frac{1}{1 + \frac{2}{n} \sum_{v=1}^{n} \frac{1}{K_v - 1}} \right) \|p\|.$$

This is clearly an improvement of Theorem 2.3 and reduces to it if $K_v = 1$ for some $v = 1, 2, \ldots, n$ and to Theorem 2.4 if $K_v \geq K \geq 1$ for $v = 1, 2, \ldots, n$. The proof of Theorem 2.6 employs the following lemma:

Lemma 2.2 *If $p(z) = a_n \Pi_{v=1}^{n}(z - z_v)$, $|z_v| \geq K_v \geq 1$, and $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ then for $|z| = 1$*

$$t_0 |p'(z)| \leq |q'(z)|$$

where

$$t_0 = \frac{\sum_{v=1}^{n} \frac{K_v}{K_v - 1}}{\sum_{v=1}^{n} \frac{1}{K_v - 1}}.$$

Theorems 2.3 through 2.6 can be collected into one result. We will prove this result and obtain these other theorems as corollaries. We need the following lemma due to Shapiro [58]:

Lemma 2.3 *Let $\mathcal{P}_n$ denote the linear space of polynomials $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ of degree $\leq n$ with complex coefficients, normed by $\|p\| = \max_{0 \leq \theta < 2\pi} |p(e^{i\theta})|$. Define the linear functional L on $\mathcal{P}_n$ as $L : p \to l_0 a_0 + l_1 a_1 + \cdots + l_n a_n$ where the l_i are complex numbers. If the norm of the functional is N then*

$$\int_0^{2\pi} \Theta\left(\frac{\left|\sum_{k=0}^{n} l_k a_k e^{ik\theta}\right|}{N}\right) d\theta \leq \int_0^{2\pi} \Theta\left(\left|\sum_{k=0}^{n} a_k e^{ik\theta}\right|\right) d\theta$$

where $\Theta(t)$ is a nondecreasing convex function of t.

Our result is:

Theorem 2.7 *If $p(z) = a_n \Pi_{v=1}^{n}(z - z_v)$ and $|z_v| \geq K_v \geq 1$ for $v = 1, 2, \ldots, n$, then for $\delta > 1$,*

$$\|p'\|_\delta \leq n E_\delta \|p\|_\delta$$

where

$$E_\delta = \left\{\frac{1}{2\pi} \int_0^{2\pi} |t_0 + e^{i\alpha}|^\delta d\alpha\right\}^{-1/\delta}$$

and

$$t_0 = \left(\frac{\sum_{v=1}^{n} \frac{K_v}{K_v - 1}}{\sum_{v=1}^{n} \frac{1}{K_v - 1}} \right) = 1 + \frac{1}{\sum_{v=1}^{n} \frac{1}{K_v - 1}}.$$

Proof. If $|p(z)| \le M$ for $|z| \le 1$ then for $|\lambda| > 1$ the polynomial $P(z) = p(z) - \lambda M$ does not vanish in $|z| \le 1$. Let

$$Q(z) \equiv z^n \overline{P\left(\frac{1}{\overline{z}}\right)} = z^n \overline{p\left(\frac{1}{\overline{z}}\right)} - z^n \overline{\lambda} M \equiv q(z) - z^n \overline{\lambda} M$$

where $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$. The polynomial $Q(z)$ has all its zeros in $|z| < 1$ and $|Q(z)| = |P(z)|$ for $|z| = 1$. So, by Lemma 2.1 it follows that $|P'(z)| \le |Q'(z)|$ for $|z| = 1$. Thus for $0 \le \theta \le 2\pi$,

$$|P'(z)| = |p'(z)| = \left| \frac{dp(e^{i\theta})}{d\theta} \right| \le |Q'(z)| = \left| \frac{dq(e^{i\theta})}{d\theta} - ine^{in\theta} \overline{\lambda} M \right|.$$

By choosing $arg(\lambda)$ suitably, we obtain

$$\left| \frac{dp(e^{i\theta})}{d\theta} \right| \le n|\lambda| M - \left| \frac{dq(e^{i\theta})}{d\theta} \right|.$$

Now, letting $|\lambda| \to 1$ we get

$$\left| \frac{dp(e^{i\theta})}{d\theta} \right| + \left| \frac{dq(e^{i\theta})}{d\theta} \right| \le Mn \tag{2.1}$$

But

$$\left| \frac{dq(e^{i\theta})}{d\theta} \right| = \left| \frac{d\{e^{-in\theta} p(e^{i\theta})\}}{d\theta} \right| = \left| -inp(e^{i\theta}) + \frac{dp(e^{i\theta})}{d\theta} \right|$$

and therefore by (2.1),

$$\left|\frac{dp(e^{i\theta})}{d\theta}\right| + \left|-inp(e^{i\theta}) + \frac{dp(e^{i\theta})}{d\theta}\right| \le Mn.$$

Hence for every α such that $0 \le \alpha < 2\pi$ we have by the triangle inequality

$$\left|\frac{dp(e^{i\theta})}{d\theta} + e^{i\alpha}\left\{-inp(e^{i\theta}) + \frac{dp(e^{i\theta})}{d\theta}\right\}\right| \le Mn$$

or

$$\left|(e^{i\alpha}+1)\frac{dp(e^{i\theta})}{d\theta} - ine^{i\alpha}p(e^{i\theta})\right| \le Mn.$$

Thus the norm of the bounded linear functional:

$$L : p \to \left[(e^{i\alpha}+1)\frac{dp(e^{i\theta})}{d\theta} - ine^{i\alpha}p(e^{i\theta})\right]_{\theta=0}$$

is a number satisfying $N \le n$, and it follows from Lemma 2.3 that for every $\delta \ge 1$,

$$\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta} + e^{i\alpha}\left\{-inp(e^{i\theta}) + \frac{dp(e^{i\theta})}{d\theta}\right\}\right|^\delta d\theta \le N^\delta \int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta \qquad (2.2)$$

which implies

$$\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta} + e^{i\alpha}\left\{-inp(e^{i\theta}) + \frac{dp(e^{i\theta})}{d\theta}\right\}\right|^\delta d\theta \le n^\delta \int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta. \qquad (2.3)$$

By Lemma 2.2,

$$t_0\left|\frac{dp(e^{i\theta})}{d\theta}\right| \le \left|\frac{dq(e^{i\theta})}{d\theta}\right| = \left|-inp(e^{i\theta}) + \frac{dp(e^{i\theta})}{d\theta}\right| \qquad (2.4)$$

where t_0 is as in Lemma 2.2. Now,

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta}+e^{i\alpha}\left\{-inp(e^{i\theta})+\frac{dp(e^{i\theta})}{d\theta}\right\}\right|^{\delta}d\theta\,d\alpha$$

$$=\int_0^{2\pi}\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta}\right|\left|1+e^{i\alpha}\left(\frac{-inp(e^{i\theta})+\frac{dp(e^{i\theta})}{d\theta}}{\frac{dp(e^{i\theta})}{d\theta}}\right)\right|^{\delta}d\theta\,d\alpha$$

inverting the order of integration,

$$\begin{aligned}&=\int_0^{2\pi}\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta}\right|^{\delta}\left|1+e^{i\alpha}\left(\frac{-inp(e^{i\theta})+\frac{dp(e^{i\theta})}{d\theta}}{\frac{dp(e^{i\theta})}{d\theta}}\right)\right|^{\delta}d\alpha\,d\theta\\&=\int_0^{2\pi}\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta}\right|^{\delta}\left|1+e^{i\alpha}\frac{B(e^{i\theta})}{A(e^{i\theta})}\right|^{\delta}d\alpha\,d\theta \qquad (2.5)\end{aligned}$$

where $t_0|A(e^{i\theta})|\leq|B(e^{i\theta})|$ by (2.4) and so $\left|\frac{B(e^{i\theta})}{A(e^{i\theta})}\right|\geq t_0$. Thus for every fixed θ and every $\delta\geq 1$,

$$\int_0^{2\pi}\left|1+\left|\frac{B(e^{i\theta})}{A(e^{i\theta})}\right|e^{i\alpha}\right|^{\delta}d\alpha\geq\int_0^{2\pi}|1+t_0e^{i\alpha}|^{\delta}d\alpha. \qquad (2.6)$$

This follows, for example, by taking $G(\zeta)=\frac{1}{t_0}+\zeta$ and $R=\left|\frac{B(e^{i\theta})}{A(e^{i\theta})}\right|$ in the well known inequality

$$\int_0^{2\pi}|G(Re^{i\alpha})|^{\delta}d\alpha\geq\int_0^{2\pi}|G(e^{i\alpha})|^{\delta}d\alpha$$

where $\delta>0$ and $R\geq 1$, valid for every entire function $G(\zeta)$. Now,

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta}\right|^{\delta}|1+t_0e^{i\alpha}|^{\delta}d\alpha\,d\theta=\int_0^{2\pi}\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta}\right|^{\delta}|e^{-i\alpha}+t_0|^{\delta}d\alpha\,d\theta$$

$$\begin{aligned}
&= \int_0^{2\pi}\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta}\right|^\delta |t_0+e^{i\alpha}|^\delta d\alpha\, d\theta \\
&\le \int_0^{2\pi}\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta}\right|^\delta \left|1+e^{i\alpha}\frac{B(e^{i\theta})}{A(e^{i\theta})}\right|^\delta d\alpha\, d\theta && \text{by (2.6)} \\
&= \int_0^{2\pi}\int_0^{2\pi}\left|\frac{dp(e^{i\theta})}{d\theta}+e^{i\alpha}\left\{-inp(e^{i\theta})+\frac{dp(e^{i\theta})}{d\theta}\right\}\right|^\delta d\theta\, d\alpha && \text{by (2.5)} \\
&\le 2\pi n^\delta \int_0^{2\pi}|p(e^{i\theta})|^\delta d\theta && \text{by (2.3).}
\end{aligned}$$

And so,

$$\int_0^{2\pi}|p'(e^{i\theta})|^\delta d\theta \le E_\delta^\delta n^\delta \int_0^{2\pi}|p(e^{i\theta})|^\delta d\theta$$

where

$$E_\delta = \left\{\frac{1}{2\pi}\int_0^{2\pi}|t_0+e^{i\alpha}|^\delta d\alpha\right\}^{-1/\delta}.$$

□

Theorem 2.7 reduces to Theorem 2.5 if $K_v = 1$ for some $v = 1, 2, \ldots, n$. It is our belief that if $\delta \to \infty$ then Theorem 2.7 reduces to Theorem 2.6.

Chapter 3

Rate of Growth Results

In this chapter we study problems relating the the size of $\max_{|z|=r>0} |p(z)|$ in terms of $\max_{|z|=1} |p(z)|$. We start with a definition.

Definition 3.1 *For the function $f(z)$ analytic in $|z| \leq R$, define*

$$M(f, R) = \max_{|z|=R} |f(z)|$$

for $R \geq 0$, and denote $M(f, 1) \equiv \|f\|$.

From the Maximum Modulus Theorem (see, for example, p. 134 of Ahlfors [1]), we see that for a polynomial $p(z)$ (or any other entire function), $M(p, R)$ is a strictly increasing function of R and is defined for $R \in [0, \infty)$. We are concerned with the rate of growth of this function.

The first result, due to S. Bernstein, is a simple deduction from the Maximum Modulus Theorem (see [54] or Vol. 1, p. 137 of [49]):

Theorem 3.1 *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then*

$$M(p, R) \leq R^n \|p\|$$

for $R \geq 1$.

The following result which refines Theorem 3.1 is due to Rahman [52].

Theorem 3.2 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n*, then*

$$M(p,R) \leq R^n\|p\| - \frac{(\|p\| - |a_n|)(R-1)}{|a_n| + R\|p\|} R^n\|p\|$$

for $R \geq 1$*.*

In both Theorems 3.1 and 3.2, equality holds only if $P(z) = \lambda z^n$.

Another refinement of Theorem 3.1 is due to Frappier, Rahman, and Ruscheweyh [16]. The proof of the theorem is rather novel, appealling to the *Hadamard product* or *convolution* of analytic functions (see also [56]) and several results from matrix analysis.

Theorem 3.3 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n*, then*

$$R^n\|p\| - (R^n - R^{n-2})|a_n| \leq M(p,R) \leq R^n\|p\| - (R^n - R^{n-2})|a_0|$$

for $R \geq 1$*.*

Again, equality for the outer bound holds for $p(z) = \lambda z^n$. Since this polynomial has all its zeros at $z = 0$, it follows that, if we put a restriction on the location of the zeros of the polynomial, we should be able to sharpen the above results.

The first result in this direction is due to Ankeney and Rivlin [2].

Theorem 3.4 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *and* $p(z) \neq 0$ *for* $|z| < 1$, *then*

$$M(p, R) \leq \frac{R^n + 1}{2} \|p\|$$

for $R \geq 1$.

A refinement of Theorem 3.4 was given by Govil [20]. The proof is dependent on Theorems 2.3 and 3.3.

Theorem 3.5 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$, $n > 2$, *and* $p(z) \neq 0$ *for* $|z| < 1$ *then*

$$M(p, R) \leq \left(\frac{R^n + 1}{2}\right) \|p\| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2}\right) |a_1|$$

for $R \geq 1$.

Equality holds in Theorems 3.4 and 3.5 only for $p(z) = \lambda + \mu z^n$ where $|\lambda| = |\mu|$, that is, when all the zeros of $p(z)$ lie on $|z| = 1$. If more is known about the moduli of the zeros of the polynomial, Theorem 3.5 can be further refined. Govil [22], in this direction, proved:

Theorem 3.6 *If* $p(z) = a_n \Pi_{v=1}^{n} (z - z_v) = \sum_{v=1}^{n} a_v z^v$, $a_n \neq 0$, $n \geq 2$, *and* $|z_v| \geq K_v \geq 1$ *for* $v = 1, 2, \ldots, n$ *then*

$$M(p, R) \leq \left(\frac{R^n + 1}{2}\right) \left[1 - \left(\frac{R^n - 1}{R^n + 1}\right) \frac{1}{1 + \frac{2}{n} \sum_{v=1}^{n} \frac{1}{K_v - 1}}\right] \|p\|$$

$$- \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |a_1|$$

if $n > 2$, *and*

$$M(p, R) \leq \left(\frac{R^2 + 1}{2} \right) \left[1 - \left(\frac{R^2 - 1}{R^2 + 1} \right) \frac{(K_1 - 1)(K_2 - 1)}{(K_1 K_2 - 1)} \right] \|p\| - \frac{(R-1)^2}{2} |a_1|$$

if $n = 2$.

Notice that if $K_v = 1$ for some $v = 1, 2, \ldots, n$ then Theorem 3.6 reduces to Theorem 3.5.

Yet another refinement of Theorem 3.4 was given, again, by Govil [21].

Theorem 3.7 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *and* $p(z) \neq 0$ *for* $|z| < 1$ *then*

$$\begin{aligned} M(p, R) \quad &\leq \quad \left(\frac{R^n + 1}{2} \right) \|p\| - \frac{n}{2} \left(\frac{\|p\|^2 - 4|a_n|^2}{\|p\|} \right) \left\{ \frac{(R-1)\|p\|}{\|p\| + 2|a_n|} \right. \\ &\qquad \left. - \log \left(1 + \frac{(R-1)\|p\|}{\|p\| + 2|a_n|} \right) \right\} \end{aligned}$$

for $R \geq 1$.

The bound given in Theorem 3.7 is always sharper than the bound given in Theorem 3.4, except in the case $|a_n| = \frac{\|p\|}{2}$. The proof uses, besides other results, the following lemma, which can be verified by the first derivative test.

Lemma 3.1 *For $R \geq 1$ and $n \geq 0$,*

$$f(x) = \left\{1 - \frac{(x - n|a_n|)(R-1)}{n|a_n| + Rx}\right\} x$$

is an increasing function of x for $x > 0$.

We now sharpen Theorem 3.7, introducing more information on the moduli of the zeros. We need the following lemma:

Lemma 3.2 *Let $p(z) = \Pi_{v=1}^{n} a_n(z - z_v) = \sum_{v=1}^{n} a_v z^v$ and $|z_v| \geq K_v \geq 1$ for $v = 1, 2, \ldots, n$. Then*

$$|a_n| \leq \kappa \|p\|$$

where $\kappa = \dfrac{\sum_{v=1}^{n} \frac{1}{K_v - 1}}{\sum_{v=1}^{n} \frac{K_v+1}{K_v-1}} = \dfrac{1}{2}\left(1 - \dfrac{1}{1 + \frac{2}{n}\sum_{v=1}^{n} \frac{1}{K_v-1}}\right).$

Proof. Well, $p'(z) = \sum_{v=1}^{n} v a_v z^{v-1}$ and by a result due to Visser (Lemma 3 of [62]), $|a_1| + |na_n| \leq \|p'\|$ which implies in particular that $|na_n| \leq \|p'\|$. From Theorem 2.6,

$$|na_n| \leq \|p'\| \leq n\kappa\|p\|$$

and the result follows. □

Here, we prove the following generalization of Theorem 3.7

Theorem 3.8 *If $p(z) = a_n \Pi_{v=1}^{n}(z - z_v)$ and $|z_v| \geq K_v \geq 1$ for $v = 1, 2, \ldots, n$ then*

$$M(p, R) \leq \kappa\|p\|\left(R^n - 1 + \frac{1}{\kappa}\right) - n\kappa\frac{\|p\|^2 - \frac{1}{\kappa^2}|a_n|^2}{\|p\|}\left\{\frac{(R-1)\kappa\|p\|}{|a_n| + \kappa\|p\|}\right.$$

$$-\log\left(1+\frac{(R-1)\kappa\|p\|}{|a_n|+\kappa\|p\|}\right)\Bigg\}$$

for $R \geq 1$ *where* $\kappa = \dfrac{\sum_{v=1}^{n}\frac{1}{K_v-1}}{\sum_{v=1}^{n}\frac{K_v+1}{K_v-1}} = \dfrac{1}{2}\left(1+\dfrac{1}{1+\frac{2}{n}\sum_{v=1}^{n}\frac{1}{K_v-1}}\right)$.

Proof. Well,

$$\int_1^R p'(re^{i\phi})e^{i\phi}\,dr = p(Re^{i\phi}) - p(e^{i\phi})$$

so

$$\begin{aligned}
|p(Re^{i\phi}) - p(e^{i\phi})| &= \left|\int_1^R p'(re^{i\phi})e^{i\phi}\,dr\right| \\
&\leq \int_1^R |p'(re^{i\phi})|\,dr \\
&\leq \int_1^R r^{n-1}\left\{1-\frac{(\|p'\|-n|a_n|)(r-1)}{n|a_n|+r\|p'\|}\right\}\|p'\|\,dr. \qquad (3.1)
\end{aligned}$$

Equation (3.1) follows by applying Theorem 3.2 to $p'(z)$. From Lemma 3.1, the integrand of equation (3.1) is an increasing function of $\|p'\|$. So by Theorem 2.6,

$$\begin{aligned}
|p(Re^{i\phi}) - p(e^{i\phi})| &\leq \int_1^R r^{n-1}\left\{1-\frac{(\kappa n\|p\|-n|a_n|)(r-1)}{n|a_n|+r\kappa n\|p\|}\right\}\kappa n\|p\|\,dr \\
&= \kappa\|p\|(R^n-1) - \kappa n\|p\|(\kappa\|p\|-|a_n|)\int_1^R \frac{r^{n-1}(r-1)}{|a_n|+r\kappa\|p\|}\,dr.
\end{aligned}$$

From Lemma 3.2, $|a_n| \leq \kappa\|p\|$ so $\kappa\|p\| - |a_n| \geq 0$ and so

$$|p(Re^{i\phi}) - p(e^{i\phi})| \leq \kappa\|p\|(R^n-1) - \kappa n\|p\|(\kappa\|p\|-|a_n|) \times$$

$$\begin{aligned}
&\int_1^R \frac{r-1}{|a_n| + r\kappa\|p\|}\,dr \\
&= \kappa\|p\|(R^n-1) - \kappa n\|p\|(\kappa\|p\| - |a_n|) \times \\
&\quad \int_1^R \left(1 - \frac{|a_n| + \kappa\|p\|}{|a_n| + r\kappa\|p\|}\right) dr \\
&= \kappa\|p\|(R^n-1) - \kappa n\|p\|(\kappa\|p\| - |a_n|) \times \\
&\quad \left\{(R-1) - \frac{|a_n| + \kappa\|p\|}{\kappa\|p\|} \log \frac{|a_n| + \kappa\|p\|}{|a_n| + \kappa\|p\|}\right\} \\
&= \kappa\|p\|(R^n-1) - n\kappa \frac{\|p\|^2 - \frac{1}{\kappa^2}|a_n|^2}{\|p\|} \times \\
&\quad \left\{\frac{(R-1)\kappa\|p\|}{|a_n| + \kappa\|p\|} - \log\left(1 + \frac{(R-1)\kappa\|p\|}{|a_n| + \kappa\|p\|}\right)\right\}.
\end{aligned}$$

Notice that $|p(Re^{i\phi})| - \|p\| \leq |p(Re^{i\phi}) - p(e^{i\phi})|$. So

$$\begin{aligned}
M(p,R) \leq\ & \kappa\|p\|\left(R^n - 1 + \frac{1}{\kappa}\right) - n\kappa \frac{\|p\|^2 - \frac{1}{\kappa^2}|a_n|^2}{\|p\|} \left\{\frac{(R-1)\kappa\|p\|}{|a_n| + \kappa\|p\|}\right. \\
& \left. - \log\left(1 + \frac{(R-1)\kappa\|p\|}{|a_n| + \kappa\|p\|}\right)\right\}.
\end{aligned}$$

□

As a corollary, if we only know that the polynomial has no zeros in $|z| < K$ where $K \geq 1$ then we get:

Corollary 3.1 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *and* $p(z) \neq 0$ *for* $|z| < K$, $K \geq 1$, *then*

$$M(p,R) \leq \left(\frac{R^n + K}{1+K}\right)\|p\| - \frac{n}{1+K}\left(\frac{\|p\|^2 - (1+K)^2|a_n|^2}{\|p\|}\right)\left\{\frac{(R-1)\|p\|}{\|p\| + (1+K)|a_n|}\right.$$

$$-\log\left(1+\frac{(R-1)\|p\|}{\|p\|+(1+K)|a_n|}\right)\Bigg\}$$

for $R \geq 1$.

If we let $K = 1$ in Corollary 3.1 then we get Theorem 3.7.

We now shift our attention to the behavior of $M(p, r)$ for $r \leq 1$. From the Maximum Modulus Theorem, we immediately have that $\|p\| \geq M(p, r) \geq |a_0|$ for $r \leq 1$. The first nontrivial result in this direction is:

Theorem 3.9 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *then*

$$M(p, r) \geq r^n \|p\|$$

for $r \leq 1$.

The proof of Theorem 3.9 is implicit in [49] (see p. 137 problem 269) and the theorem can be credited to Bernstein. The theorem is explicitly stated and proved by Varga [61] and a simple proof is also presented by Qazi [50].

Another result in this direction is due to Frappier, Rahman and Ruscheweyh [16]. This theorem is related to Theorem 3.3 and the method of proof is similar.

Theorem 3.10 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *then*

$$\|p\| - (1 - r^2)|a_n| \geq M(p, r) \geq r^n \|p\| + (1 - r^2)|a_0|$$

for $r \le 1$.

We present a related result:

Theorem 3.11 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *then*

$$M(p,r) \ge \frac{1}{2} r^{n-1}(\|p\| - |a_n|) + \frac{1}{2}\sqrt{r^{2n-2}(\|p\| - |a_n|)^2 + 4r^{2n}|a_n|\|p\|}$$

for $r \le 1$.

Proof: Let $R = \dfrac{1}{r}$. Then

$$\begin{aligned}
\|p\| &= \max_{|z|=R}\left|p\left(\frac{z}{R}\right)\right| \\
&\le R^n \left\{1 - \frac{\left(M(p,r) - \frac{|a_n|}{R^n}\right)(R-1)}{RM(p,r) + \frac{|a_n|}{R^n}}\right\} M(p,r) \qquad \text{from Theorem 3.2} \\
&= R^n \left\{\frac{R|a_n| + R^n M(p,r)}{R^{n+1}M(p,r) + |a_n|}\right\} M(p,r).
\end{aligned}$$

So,

$$\|p\|(R^{n+1}M(p,r) + |a_n|) \le \left(R^{n+1}|a_n| + R^{2n}M(p,r)\right) M(p,r)$$

or

$$R^{2n}M(p,r)^2 + \left(R^{n+1}|a_n| - R^{n+1}\|p\|\right) M(p,r) - |a_n|\|p\| \ge 0.$$

Therefore,

$$M(p,r) \ \ge \ \frac{R^{n+1}(\|p\| - |a_n|) + \sqrt{R^{2n+2}(|a_n| - \|p\|)^2 + 4R^{2n}|a_n|\|p\|}}{2R^{2n}}$$

$$= \frac{1}{2}r^{n-1}(\|p\| - |a_n|) + \frac{1}{2}\sqrt{r^{2n-2}(|a_n| - \|p\|)^2 + 4r^{2n}|a_n|\|p\|}.$$

□

We suspect that the bound given in Theorem 3.11 is, in general, an improvement of the bound given in Theorem 3.10, however we have not been able to show this. By means of the following example, we show that in some cases the bound obtained in Theorem 3.11 can be considerably better than the bound obtained from Theorem 3.10.

Example 3.1 *If* $p(z) = (20-20i)z+(30-50i)z^2+(1+5i)z^3+.0001z^4$ *then Theorems 3.8 and 3.10 both yield the result that* $M(p, .5) \geq 5.164$*. Theorem 3.11 gives that* $M(p, .5) \geq 10.329$*, which is an improvement of a factor of* 2*.*

Theorems 3.10 and 3.11 put no requirements on $p(z)$. With a restriction on the location of the zeros of $p(z)$, we should be able to sharpen the above bounds for certain classes of polynomials. The first result in this direction is due to Rivlin [55]:

Theorem 3.12 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *and* $p(z) \neq 0$ *for* $|z| < 1$ *then*

$$M(p, r) \geq \left(\frac{1+r}{2}\right)^n \|p\|$$

for $0 < r < 1$*.*

Equality holds for $p(z) = (\lambda + \mu z)^n$ where $|\lambda| = |\mu|$. Govil [19] offered a generalization of Theorem 3.12, allowing a comparison of $M(p, r_1)$ and $M(p, r_2)$ where $0 \leq r_1 < r_2 \leq 1$. Namely, he proved:

Theorem 3.13 *If* $p(z) = \sum_{v=0}^{n} a_v z^v$, $p(z) \neq 0$ *for* $|z| < 1$ *and* $p'(0) = 0$ *then*

$$M(p, r_1) \geq \left(\frac{1 + r_1}{1 + r_2}\right)^n \left\{ \frac{1}{1 - \frac{(1-r_2)(r_2-r_1)n}{4} \left(\frac{1+r_1}{1+r_2}\right)^{n-1}} \right\} M(p, r_2)$$

for $0 \leq r_1 < r_2 \leq 1$.

Notice for $r_2 = 1$, Theorem 3.13 implies Theorem 3.12.

Chapter 4

Results for Entire Functions

In this chapter, we obtain results concerning extremal problems for functions of exponential type. Our results will generalize some of the results of Chapters 2 and 3. We start with some definitions.

Definition 4.1 *For an entire function $f(z)$, define $M(f,r) = \max_{|z|=r} |f(z)|$. The order ρ of $f(z)$ is $\rho = \limsup_{r\to\infty} \frac{\log\log M(f,r)}{\log r}$. An entire function of positive order ρ is of type τ if $\tau = \limsup_{r\to\infty} \frac{\log M(f,r)}{r^\rho}$.*

Definition 4.2 *An entire function is of exponential type τ if either it is of order 1 and type less than or equal to τ, or it is of order less than 1. Denote the class of entire functions of exponential type τ by $\mathcal{E}_\tau$.*

Definition 4.3 *For $f \in \mathcal{E}_\tau$, let $\|f\| = \sup_{x\in\mathbf{R}} |f(x)|$. Define the indicator function to be $h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}$.*

The following elementary examples illustrate the idea of order and type.

Example 4.1 *The function $f(z) = e^{\tau z^\rho}$ is of order ρ and type τ.*

Example 4.2 *Suppose $f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2}$. Then*

$$\frac{e^r - 1}{2} \le \max_{|z|=r} |\sin z| \le \frac{e^r + 1}{2}.$$

So $f(z) = \sin z$ is of order $\rho = 1$ and type $\tau = 1$.

For further properties of entire functions, see Holland [28] and Markushevich [42], which offer nice introductions. The classical (and in depth) text on the subject is due to Boas [7]. This book deals at length with entire functions of exponential type and we will quote this reference often. Additional references with introductory material as well as applications are Paley and Wiener [47], Levinson [38], Levin [37] and Young [63].

The first result dealing with the norm of the derivative of an entire function of exponential type is due to Bernstein [6]. He proved:

Theorem 4.1 *If $f \in \mathcal{E}_\tau$ and $\|f\| = M$, then $\|f'\| \leq M\tau$.*

This is quite clearly the analogue of Theorem 2.2 for entire functions of exponential type. In fact, we will find that the results for $\mathcal{P}_n$ which extend to $\mathcal{E}_\tau$ will, in some sense, have τ corresponding to n. Additionally, if we consider the transformation $T : z \to e^{iz}$ then $T : \mathbf{R} \to \{z : |z| = 1\}$ and $T : \{z : Im(z) > 0\} \to \{z : |z| < 1\}$. The transformation T also reinforces the belief that there is a correspondence between τ and n when we notice that if $p(z) \in \mathcal{P}_n$ then $p(e^{iz}) \in \mathcal{E}_n$. Also, as hinted at by T, theorems which restrict the location of the zeros of polynomials to the exterior of the unit disc, will have analogues for entire functions of exponential type which restrict the location of the zeros to the lower half plane. For example, the following result due to Boas [9] is a generalization of Theorem 2.3.

Theorem 4.2 *If $f \in \mathcal{E}_\tau$, $\|f\| = M$, $h_f\left(\frac{\pi}{2}\right) = 0$ and $f(z) \neq 0$ for $Im(z) > 0$, then*

$$\|f'\| \leq \frac{1}{2} M\tau.$$

In addition to bounds on the norm of the derivative, we can put bounds on the rate of growth of $|f(z)|$ as a function of $Im(z)$, similar to the theorems of Chapter 2. The following result is also due to Boas [9].

Theorem 4.3 *For $f \in \mathcal{E}_\tau$, $\|f\| = M$, $h_f\left(\frac{\pi}{2}\right) = 0$ and $f(z) \neq 0$ for $Im(z) > 0$, we have*

$$|f(z)| \leq \frac{M}{2}\left(e^{\tau|y|} + 1\right)$$

where $y = Im(z) \leq 0$.

Clearly, Theorem 4.3 generalizes Theorem 3.4.

Let us again consider results for polynomials, including a generalization of the differentiation operator.

Definition 4.4 *If $p \in \mathcal{P}_n$, then define the polar derivative of p with respect to a complex number ζ to be*

$$D_\zeta[p] = np(z) - (z - \zeta)p'(z).$$

The zeros of $D_\zeta[p]$ are invariant under the general linear transformation $z \to \dfrac{\alpha z + \beta}{\gamma z + \delta}$ where $\alpha\delta - \beta\gamma \neq 0$ (see Marden [40] for this and related results). Concerning the location of the zeros of $D_\zeta[p]$, Laguerre [34] proved:

Theorem 4.4 *Let $p \in \mathcal{P}_n$. If $p(z) \neq 0$ in a closed or open circular domain K, then $D_\zeta[p]$ is nonzero for $z \in K$ and $\zeta \in K$. (By closed "circular domain" is meant the closed interior or closed exterior of a circle or half-plane.)*

Notice that if we divide $D_\zeta[p]$ by ζ and make $\zeta \to \infty$, then we get by Theorem 4.4 that $p'(z) \neq 0$ for $z \in K$. This is the well known Gauss-Lucas Theorem (see, for example, p. 29 of Ahlfors [1]).

Aziz [3] presented a rate of growth bound on the k^{th} polar derivative of a polynomial. He applied Theorem 4.4 to prove:

Theorem 4.5 *Let $p \in \mathcal{P}_n$, $\|p\| = 1$ and $p(z) \neq 0$ for $|z| < 1$. Then for $|z| \geq 1$,*

$$|D_{\zeta_1} \cdots D_{\zeta_k}[p(z)]| \leq \frac{n(n-1)\cdots(n-k+1)}{2} \left\{ |\zeta_1 \cdots \zeta_k z^{n-k}| + 1 \right\}$$

where $|\zeta_i| \geq 1$ for $i = 1, 2, \ldots, k$.

Aziz also put a bound on the norm of $D_\zeta[p]$ for $p \in \mathcal{P}_n$. Again applying Thoerem 4.4, he proved:

Theorem 4.6 *Let $p \in \mathcal{P}_n$, $\|p\| = 1$ and $p(z) \neq 0$ for $|z| < k$ where $k \geq 1$. Then for $|\zeta| \geq 1$,*

$$\|D_\zeta[p]\| \leq n \left(\frac{k + |\zeta|}{1 + k} \right).$$

We now proceed to obtain results for entire functions of exponential type which will generalize some of the results of Aziz. This will require an extension of Theorem 4.4 to the space $\mathcal{E}_\tau$. We start with a definition which is due to Rahman and Schmeisser [53].

Definition 4.5 *If $f \in \mathcal{E}_\tau$ then define the polar derivative of f with respect to a complex number ζ to be*

$$D_\zeta[f] = \tau f(z) + i(1 - \zeta) f'(z).$$

In the following, by "open upper half plane" we mean $\{z|Im(z)>0\}$. The "open lower half plane" is similarly defined. Rahman and Schmeisser [53] were successful in using Definition 4.5 to get the extension of Theorem 4.4:

Theorem 4.7 *Let $f \in \mathcal{E}_\tau$ where $\tau > 0$ and $h_f\left(\frac{\pi}{2}\right) = 0$. Let H denote the (open or closed) upper half plane. If $f(z) \neq 0$ for $z \in H$, then $D_\zeta[f(z)] \neq 0$ for $z \in H$ and $|\zeta| \leq 1$.*

Since we wish to let $\zeta \to \infty$, we prove the following result, which will be needed later.

Lemma 4.1 *Let $f \in \mathcal{E}_\tau$ where $\tau > 0$ and $h_f\left(\frac{-\pi}{2}\right) = \tau$. Let L denote the (open or closed) lower half plane. If $f(z) \neq 0$ for $z \in L$, then $D_\zeta[f(z)] \neq 0$ for $z \in L$ and $|\zeta| \geq 1$.*

Proof. Let $g(z) = e^{i\tau z}\overline{f(\overline{z})}$. Then $g(z) \neq 0$ for $z \in H$. Also

$$h_g\left(\frac{\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|e^{-\tau y}\overline{f(-iy)}|}{y} = -\tau + h_f\left(\frac{-\pi}{2}\right) = 0.$$

So, applying Theorem 4.7 to $g(z)$,

$$D_\zeta[g] = \tau g(z) + i(1-\zeta)g'(z) \neq 0 \tag{4.1}$$

for $z \in H$ and $|\zeta| \leq 1$. Since

$$g'(z) = i\tau e^{i\tau z}\overline{f(\overline{z})} + e^{i\tau z}\overline{f'(\overline{z})},$$

equation (4.1) is equivalent to

$$\tau e^{i\tau z}\overline{f(\overline{z})} + i(1-\zeta)\left(i\tau e^{i\tau z}\overline{f(\overline{z})} + e^{i\tau z}\overline{f'(\overline{z})}\right) = e^{i\tau z}\left(\zeta\tau\overline{f(\overline{z})} + i(1-\zeta)\overline{f'(\overline{z})}\right) \neq 0$$

for $z \in H$ and $|\zeta| \leq 1$. This means

$$\zeta\tau\overline{f(\overline{z})} + i(1-\zeta)\overline{f'(\overline{z})} \neq 0$$

which gives

$$\tau\overline{f(\overline{z})} + i\left(\frac{1}{\zeta} - 1\right)\overline{f'(\overline{z})} \neq 0$$

that is

$$\tau f(\overline{z}) + i\left(1 - \frac{1}{\zeta}\right) f'(\overline{z}) \neq 0$$

which implies

$$\tau f(\overline{z}) + i\left(1 - \frac{\zeta}{|\zeta|^2}\right) f'(\overline{z}) \neq 0$$

for $z \in H$ and $|\zeta| \leq 1$. Now if $z \in H$ then $\overline{z} \in L$ and if $|\zeta| \leq 1$ then $\left|\frac{\zeta}{|\zeta|^2}\right| = \frac{1}{|\zeta|} \geq 1$. So

$$\tau f(z) + i(1-\zeta)f'(z) \neq 0$$

for $z \in L$ and $|\zeta| \geq 1$. □

Letting $\zeta \to \infty$, we get that if $f(z)$ is an entire function of type τ, $h_f\left(\frac{\pi}{2}\right) = \tau$, $f(z) \neq 0$ in L, then $f'(z) \neq 0$ for $z \in L$, which corresponds to the Gauss-Lucas Theorem.

We will now extend Theorem 4.6 to functions of exponential type. First, we will need several preliminary results. As a consequence of the Phragmen-Lindelöf Theorem (see, for example, p. 3 of [7]), we have (6.2.4 of [7], also [14], [48], and [49]):

Lemma 4.2 *If $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) \leq c$ and $\|f\| \leq M$, then for every z with $y = Im(z) \geq 0$,*

$$|f(z)| \leq Me^{cy}.$$

This is an extension of the Maximum Modulus Theorem (see, for example, p. 134 of [1]) in that we get the boundedness of a function inside an unbounded region from the hypothesis that the function is bounded on the boundary and not of too rapid growth inside.

A more general result similar to Lemma 4.2 is (see [8] and p. 82 of [7]):

Lemma 4.3 *If $f \in \mathcal{E}_\tau$ and $\|f\| = M$, then for $y = Im(z)$,*

$$|f(z)| \leq Me^{\tau|y|}.$$

A result concerning the indicator function is the following:

Lemma 4.4 *If f is an entire function of order 1 and type τ, $\|f\| = M$ and $h_f\left(\frac{\pi}{2}\right) \leq 0$, then $h_f\left(\frac{-\pi}{2}\right) = \tau$.*

Proof. Let $g(z) = e^{-i\tau z/2}f(z)$. Then

$$h_g\left(\frac{\pi}{2}\right) = \frac{\tau}{2} + h_f\left(\frac{\pi}{2}\right) \leq \frac{\tau}{2}$$

and

$$h_g\left(\frac{-\pi}{2}\right) = \frac{-\tau}{2} + h_f\left(\frac{-\pi}{2}\right) \leq \frac{\tau}{2}.$$

So, from a result of Boas (see p. 82, line 14 of [7]), since $\|g\|$ is bounded and $h_g\left(\pm\frac{\pi}{2}\right) \leq \frac{\tau}{2}$, we have

$$|h_g(\theta)| \leq \frac{\tau}{2}|\sin\theta| \text{ for all } \theta.$$

So

$$\begin{aligned} h_g(\theta) &= \limsup_{R\to\infty} \frac{\log|g(Re^{i\theta})|}{R} \\ &= \limsup_{R\to\infty} \frac{\log\left|e^{-i\tau Re^{i\theta}/2} f(Re^{i\theta})\right|}{R} \\ &= \limsup_{R\to\infty} \frac{\log|e^{-i\tau Re^{i\theta}/2}|}{R} + h_f(\theta) \\ &\geq \frac{-\tau}{2} + h_f(\theta). \end{aligned}$$

So,

$$\frac{\tau}{2} \geq \frac{\tau}{2}|\sin\theta| \geq \frac{-\tau}{2} + h_f(\theta)$$

and $h_f(\theta) < \tau$ for $\theta \neq \pm\frac{\pi}{2}$. But $h_f\left(\frac{\pi}{2}\right) \leq 0$ and f is type τ implies that $h_f\left(\frac{-\pi}{2}\right) = \tau$.

$\square$

Concerning functions with no zeros in the upper half plane, we need the following result originally due to Levin [36] (see also Theorem 7.8.1 of Boas [7]).

Lemma 4.5 *Let $f \in \mathcal{E}_\tau$, $f(z) \neq 0$ for $Im(z) > 0$ and $h_f(\alpha) \geq h_f(-\alpha)$ for some α, $0 < \alpha < \pi$. Then $|f(z)| \geq |f(\overline{z})|$ for $y = Im(z) \geq 0$.*

From Lemma 4.5 we can prove the following:

Lemma 4.6 *let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{-\pi}{2}\right) = \tau$, $h_f\left(\frac{\pi}{2}\right) \le 0$, *and* $f(z) \neq 0$ *for* $Im(z) \le 0$. *Then* $|f(z)| \ge |g(z)|$ *for* $Im(z) \le 0$ *where* $g(z) = e^{i\tau z}\overline{f(\overline{z})}$.

Proof. Let $f_1(z) = e^{i\tau z/2}\overline{f(\overline{z})}$. Then $f_1(z)$ has no zero in $Im(z) > 0$. Also,

$$h_{f_1}\left(\frac{-\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|e^{\tau y/2}\overline{f(iy)}|}{y} = \frac{\tau}{2} + h_f\left(\frac{\pi}{2}\right) \le \frac{\tau}{2},$$

and

$$h_{f_1}\left(\frac{\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|e^{-\tau y/2}\overline{f(-iy)}|}{y} = \frac{-\tau}{2} + h_f\left(\frac{-\pi}{2}\right) = \frac{\tau}{2}.$$

So $h_{f_1}\left(\frac{\pi}{2}\right) \ge h_{f_1}\left(\frac{-\pi}{2}\right)$ and by Lemma 4.5, $|f_1(z)| \ge |f_1(\overline{z})|$ for $Im(z) \ge 0$. This is equivalent to

$$|e^{i\tau z/2}\overline{f(\overline{z})}| \ge |e^{i\tau \overline{z}/2}\overline{f(z)}| \text{ for } Im(z) \ge 0,$$

which gives

$$|e^{i\tau \overline{z}/2}\overline{f(z)}| \ge |e^{i\tau z/2}\overline{f(\overline{z})}| \text{ for } Im(z) \le 0,$$

that is

$$|e^{-i\tau z/2}f(z)| \ge |e^{i\tau z/2}\overline{f(\overline{z})}| \text{ for } Im(z) \le 0$$

which implies

$$|f(z)| \ge |e^{i\tau z}\overline{f(\overline{z})}| \equiv |g(z)| \text{ for } Im(z) \le 0.$$

So $|f(z)| \ge |g(z)|$ for $Im(z) \le 0$ where $g(z) = e^{i\tau z}\overline{f(\overline{z})}$. □

We also need a result of Govil [18]:

Lemma 4.7 *Let $f(z)$ be an entire function of order 1 and type τ, $h_f\left(\frac{\pi}{2}\right) \leq 0$, $\|f\| = M$ and let $g(z) = e^{i\tau z}\overline{f(\overline{z})}$. Then the type of g is less than or equal to τ.*

Also,

Lemma 4.8 *Let $f \in \mathcal{E}_\tau$, $h_f\left(\frac{-\pi}{2}\right) = \tau$, $h_f\left(\frac{\pi}{2}\right) \leq 0$, $\|f\| = M$ and $|f(z)| \geq |g(z)|$ for $Im(z) \leq 0$ where $g(z) = e^{i\tau z}\overline{f(\overline{z})}$. Then for $|\alpha| > 1$,*

$$h_{g(z)-\alpha f(z)}\left(\frac{-\pi}{2}\right) = \tau.$$

Proof. Since $|f(z)| \geq |g(z)|$ for $Im(z) \leq 0$, we have $|f(-iy)| \geq |g(-iy)|$ for $y \geq 0$. Note that

$$\begin{aligned} |g(-iy) - \alpha f(-iy)| &\geq |\alpha f(-iy)| - |g(-iy)| \\ &= |f(-iy)|\left(|\alpha| - \left|\frac{g(-iy)}{f(-iy)}\right|\right) \text{ for } y \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} h_{g(z)-\alpha f(z)}\left(\frac{-\pi}{2}\right) &= \limsup_{y\to\infty}\left(\frac{\log|g(-iy) - \alpha f(-iy)|}{y}\right) \\ &\geq \limsup_{y\to\infty}\frac{\log\left(|f(-iy)|\left\{|\alpha| - \left|\frac{g(-iy)}{f(-iy)}\right|\right\}\right)}{y} \\ &= h_f\left(\frac{-\pi}{2}\right) + 0 \\ &= \tau, \end{aligned}$$

because $\left|\dfrac{g(-iy)}{f(-iy)}\right| \leq 1$ for $y \geq 0$. This gives

$$h_{g(z)-\alpha f(z)}\left(\frac{\pi}{2}\right) \geq \tau. \tag{4.2}$$

The facts that $f \in \mathcal{E}_\tau$ and, by Lemma 4.7, $g \in \mathcal{E}_\tau$, imply that $g(z)-\alpha f(z)$, being a linear combination of f and g, is also in $\mathcal{E}_\tau$. This implies that $h_{g(z)-\alpha f(z)}\left(\frac{\pi}{2}\right) \leq \tau$. This when combined with (4.2), gives the result. □

We need the following results for the polar derivative of entire functions of exponential type.

Lemma 4.9 *Let $f \in \mathcal{E}_\tau$ with $\|f\| = 1$ and $h_f\left(\frac{\pi}{2}\right) = 0$. Then for any z with $Im(z) = y \leq 0$ and $|\zeta| \geq 1$, we have*

$$|D_\zeta[f(z)]| + |D_\zeta[g(z)]| \leq \tau\left(|\zeta|e^{\tau|y|} + 1\right),$$

where $g(z) = e^{i\tau z}\overline{f(\overline{z})}$.

Proof. By Lemma 4.3, $|f(z)| \leq e^{\tau|y|}$ where $y = Im(z)$. Let

$$F(z) = f(z) - \beta e^{i\tau z} = f(z) - \beta e^{i\tau x}e^{-\tau y}.$$

Then

$$|f(z)| \neq |\beta e^{i\tau z}| = |\beta|e^{-\tau y} = |\beta|e^{\tau|y|}$$

for $|\beta| > 1$ and $y \leq 0$. Hence $F(z) \neq 0$ in $Im(z) \leq 0$.

We now wish to show that $h_F\left(\frac{-\pi}{2}\right) = \tau$. For this, note that

$$\begin{aligned} |F(-iy)| &= |f(-iy) - \beta e^{\tau y}| \geq |\beta| e^{\tau y} - |f(-iy)| \\ &\geq |\beta| e^{\tau y} - e^{\tau y} \text{ for } y \geq 0 \\ &= e^{\tau y}(|\beta| - 1). \end{aligned}$$

So,

$$\frac{\log|F(-iy)|}{y} \geq \tau + \frac{\log(|\beta|-1)}{y}$$

and

$$h_F\left(\frac{-\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|F(-iy)|}{y} \geq \limsup_{y\to\infty}\left(\tau + \frac{\log(|\beta|-1)}{y}\right) = \tau.$$

Also, since $f(z)$ and $\beta e^{i\tau z}$ are both of exponential type τ, then $F(z)$ is of type less than or equal to τ and so $h_F\left(\frac{-\pi}{2}\right) = \tau$. Additionally, since $h_f\left(\frac{\pi}{2}\right) = 0$, we have $|f(z)| \leq 1$ for $Im(z) \geq 0$ by Lemma 4.2. Hence for $Im(z) = y \geq 0$, we have

$$\begin{aligned} |F(iy)| &= \left|f(iy) - \beta e^{-\tau y}\right| \\ &\leq |f(iy)| + |\beta| e^{-\tau y} \\ &\leq 1 + |\beta|. \end{aligned}$$

So $h_F\left(\frac{\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|F(iy)|}{y} \leq 0.$

Now, let

$$G(z) \equiv e^{i\tau z}\overline{F(\overline{z})} = e^{i\tau z}(\overline{f(\overline{z}) - \beta e^{i\tau\overline{z}}})$$

$$= e^{i\tau z}\overline{f(\overline{z})} - \overline{\beta} \equiv g(z) - \overline{\beta}.$$

Applying Lemma 4.6 to $F(z)$, $|F(z)| \geq |G(z)|$ for $Im(z) \leq 0$. Applying Lemma 4.8 to $F(z)$, we get $h_{G(z)-\alpha F(z)}\left(\frac{-\pi}{2}\right) = \tau$ where $|\alpha| > 1$.

Applying Lemma 4.1 to $G(z) - \alpha F(z)$, which is nonzero for $Im(z) \leq 0$, we get

$$D_\zeta[G(z) - \alpha F(z)] \neq 0$$

for $Im(z) \leq 0$ and $|\zeta| \geq 1$, which implies

$$|D_\zeta[G(z)]| \leq |D_\zeta[F(z)]| \tag{4.3}$$

for $Im(z) \leq 0$ and $|\zeta| \geq 1$. For if

$$|D_\zeta[G(z_0)]| > |D_\zeta[F(z_0)]|$$

for some z_0 with $Im(z_0) \leq 0$, then by choosing

$$\alpha = \frac{D_\zeta[G(z_0)]}{D_\zeta[F(z_0)]},$$

we could get

$$D_\zeta[G(z_0) - \alpha F(z_0)] = 0,$$

a contradiction.

Similarly, since $F(z) = f(z) - \beta e^{i\tau z} \neq 0$ for $Im(z) \leq 0$ and $h_F\left(\frac{-\pi}{2}\right) = \tau$, we have again by Lemma 4.1,

$$D_\zeta[F(z)] \neq 0$$

for $Im(z) \leq 0$ and for $|\zeta| \geq 1$. Hence

$$|D_\zeta[e^{i\tau z}]| \geq |D_\zeta[f(z)]|$$

for $Im(z) \leq 0$ and $|\zeta| \geq 1$. Since

$$|D_\zeta[G(z)]| = |D_\zeta[g(z) - \overline{\beta}]| = |D_\zeta[g(z)] - \tau\overline{\beta}|,$$

we get from (4.3) that

$$\begin{aligned} |D_\zeta[g(z)]| - \tau|\beta| \leq \quad &= \quad |D_\zeta[g(z)] - \tau\beta| \\ &\leq \quad |D_\zeta[f(z) - \beta e^{i\tau z}]| \\ &= \quad |D_\zeta[\beta e^{i\tau z}]| - |D_\zeta[f(z)]| \end{aligned}$$

by the proper choice of $arg(\beta)$. So

$$\begin{aligned} |D_\zeta[f(z)]| + |D_\zeta[g(z)]| \quad &\leq \quad |D_\zeta[\beta e^{i\tau z}]| + \tau|\beta| \\ &= \quad \tau(|\zeta\beta e^{i\tau z}| + \tau|\beta|) \\ &= \quad \tau(|\zeta e^{i\tau z}| + 1) \end{aligned}$$

if we let $|\beta| \to 1$. □

We now present the analogue of Theorem 4.5 for entire functions of exponential type.

Theorem 4.8 *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $\|f\| = 1$ *and* $f(z) \neq 0$ *for* $Im(z) > 0$. *Then*

$$|D_\zeta[f(z)]| \leq \frac{\tau}{2}\left(|\zeta| e^{\tau|y|} + 1\right)$$

for $y = Im(z) \leq 0$ *and* $|\zeta| \geq 1$.

Proof. Since $f(z) \neq 0$ for $Im(z) > 0$, $g(z) = e^{i\tau z}\overline{f(\overline{z})} \neq 0$ for $Im(z) \leq 0$. Also, $\|g\| = 1$.

First, suppose f is of order 1 and type τ. By Lemma 4.4, $h_f\left(\frac{-\pi}{2}\right) = \tau$ and so

$$h_g\left(\frac{-\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|e^{\tau y}\overline{f(iy)}|}{y} = \tau + h_f\left(\frac{\pi}{2}\right) = \tau$$

and

$$h_g\left(\frac{\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|e^{-\tau y}\overline{f(-iy)}|}{y} = -\tau + h_f\left(\frac{-\pi}{2}\right) = 0.$$

Applying Lemma 4.6 to $g(z)$, gives $|g(z)| \geq |f(z)|$ for $Im(z) \leq 0$. So $|f(z)| < |\beta\, g(z)|$ for $Im(z) \leq 0$ and $|\beta| > 1$. Hence, $f(z) - \beta g(z) \neq 0$ for $Im(z) \leq 0$. Applying Lemma 4.8 to $g(z)$, we get

$$h_{f(z)-\beta g(z)}\left(\frac{-\pi}{2}\right) = \tau.$$

Applying Lemma 4.1 to $f(z) - \beta g(z)$, we get

$$D_\zeta[f(z) - \beta g(z)] \neq 0$$

for $Im(z) \leq 0$, $|\beta| > 1$ and $|\zeta| \geq 1$. So by an argument similar to the justification of equation (4.3),

$$|D_\zeta[f(z)]| \leq |D_\zeta[g(z)]|$$

for $Im(z) \leq 0$ and $|\zeta| \geq 1$. And this, when combined with Lemma 4.9, will give

$$2|D_\zeta[f(z)]| \leq \tau \left(|\zeta| e^{\tau|y|} + 1\right)$$

which means that

$$|D_\zeta[f(z)]| \leq \frac{\tau}{2} \left(|\zeta| e^{\tau|y|} + 1\right)$$

for $y = Im(z) \leq 0$ and $|\zeta| \geq 1$.

Since $\frac{\tau}{2} \left(|\zeta| e^{\tau|y|} + 1\right)$ is an increasing function of τ, the result trivially holds if f is of type less than τ. □

If we divide both sides of the conclusion of Theorem 4.8 by $|\zeta|$ and let $|\zeta| \to \infty$, then Theorem 4.8 gives $|f'(z)| \leq \frac{\tau}{2} e^{\tau|y|}$ for $Im(z) \leq 0$ and this gives, for $y = 0$, Theorem 4.2. If we let $\zeta = 1$ in Theorem 4.8, it reduces to Theorem 4.3.

We now wish to put a bound on the norm of $D_\zeta[f]$. This will extend Theorem 4.6 from polynomials to entire functions of exponential type. In this direction, Govil and Rahman [25] have put a bound on the norm of f'. They have presented the two theorems:

Theorem 4.9 *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $h_{f'}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = 1$ *and suppose* $f(z)$ *has all its zeros on* $Im(z) = K \leq 0$. *Then*

$$\|f'\| \leq \frac{\tau}{e^{c|K|} + 1}.$$

Theorem 4.10 *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $h_{f'}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = 1$ *and* $h_{g'}\left(\frac{\pi}{2}\right) \leq -c < 0$ *where* $g(z) = e^{i\tau z}\overline{f(\overline{z})}$. *Also suppose* $f(z)$ *has all its zeros in* $Im(z) \leq K \leq 0$. *Then*

$$\|f'\| \leq \frac{\tau}{e^{c|K|} + 1}.$$

Both the above theorems are best possible. We will now extend Theorems 4.9 and 4.10 to the polar derivative of $f(z)$ and get these theorems as corollaries. We need several lemmas.

Lemma 4.10 *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $\|f\| = M$ *and* $f(z) \neq 0$ *for* $Im(z) > K \geq 0$. *Then*

$$|D_\zeta[f(x)]| \leq e^{-\tau K}|D_\zeta[g(x - 2iK)]|$$

for $x \in \mathbf{R}$ *and* $|\zeta| \geq 1$ *and* $g(z) = e^{i\tau z}\overline{f(\overline{z})}$.

Proof. Let $F(z) = f(z + iK)$. Then $F(z) \neq 0$ for $Im(z) > 0$. Let $G(z) = e^{i\tau z}\overline{F(\overline{z})} = e^{-\tau K}g(z - iK)$. Then $G(z) \neq 0$ for $Im(z) < 0$ and

$$\begin{aligned} h_G\left(\frac{-\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|e^{\tau y}\overline{F(iy)}|}{y} &= \tau + h_F\left(\frac{\pi}{2}\right) \\ &= \tau + h_f\left(\frac{\pi}{2}\right) = \tau \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} h_G\left(\frac{\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|e^{-\tau y}\overline{F(-iy)}|}{y} &= -\tau + h_F\left(\frac{-\pi}{2}\right) \\ &= -\tau + h_f\left(\frac{-\pi}{2}\right) \leq 0. \end{aligned}$$

So by applying Lemma 4.6 to $G(z)$, we get $|G(z)| \geq |F(z)|$ for $Im(z) \leq 0$. So for $|\alpha| > 1$, $F(z) - \alpha G(z) \neq 0$ in $Im(z) \leq 0$. Also, applying Lemma 4.2, we see that $\|G\| \leq M$ (note that $\|G\| = \|F\| = \sup_{x \in \mathbf{R}} |f(x+iK)| \leq M$ because by Lemma 4.2 $|f(z)| \leq M$ for $Im(z) \geq 0$) and so we can apply Lemma 4.8 to $G(z)$ to get

$$h_{F(z)-\alpha G(z)}\left(\frac{-\pi}{2}\right) = \tau.$$

Applying Lemma 4.1 to $F(z) - \alpha G(z)$, we get

$$D_\zeta[F(z) - \alpha G(z)] \neq 0$$

for $Im(z) \leq 0$ and $|\zeta| \geq 1$. So by an argument similar to the justification of equation (4.3),

$$|D_\zeta[F(z)]| \leq |D_\zeta[G(z)]|$$

for $Im(z) \leq 0$ and $|\zeta| \geq 1$. In particular

$$|D_\zeta[F(x - iK)]| \leq |D_\zeta[G(x - iK)]|$$

for $x \in \mathbf{R}$. And so

$$\begin{aligned} |D_\zeta[f(x)]| &= |D_\zeta[F(x - iK)]| \\ &\leq |D_\zeta[G(x - iK)]| \end{aligned}$$

$$= \quad e^{-\tau K}|D_\zeta[g(x-2iK)]|$$

Hence,

$$|D_\zeta[f(x)]| \le e^{-\tau K}|D_\zeta[g(x-2iK)]|.$$

□

Lemma 4.11 *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{-\pi}{2}\right) = \tau$, $\|f\| = M$, *and suppose* $f(z)$ *has all its zeros in* $Im(z) \ge K \ge 0$. *Then*

$$e^{-\tau K}|D_\zeta[g(x-2iK)]| \le |D_\zeta[f(x)]|$$

for $x \in \mathbf{R}$ *and* $|\zeta| \ge 1$ *and* $g(z) = e^{i\tau z}\overline{f(\overline{z})}$.

Proof. Since $f(z)$ has all its zeros in $Im(z) \ge K \ge 0$, $g(z)$ has all its zeros in $Im(z) \le -K \le 0$. So $g(z-2iK) \ne 0$ for $Im(z) > K \ge 0$. Also,

$$h_g\left(\frac{\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log|e^{-\tau y}\overline{f(-iy)}|}{y} = -\tau + h_f\left(\frac{-\pi}{2}\right) = 0.$$

Since

$$\begin{aligned} e^{i\tau z}\overline{g(\overline{z}-2iK)} &= e^{i\tau z}\left(\overline{e^{i\tau(\overline{z}-2iK)}\overline{f(z+2iK)}}\right) \\ &= e^{i\tau z}e^{-i\tau(z+2iK)}f(z+2iK) \\ &= e^{2\tau K}f(z+2iK), \end{aligned}$$

we have, by applying Lemma 4.10 to $g(z-2iK)$, that

$$|D_\zeta[g(x-2iK)]| \leq e^{-\tau K}|D_\zeta[e^{2\tau K}\overline{f(x)}]|.$$

which gives

$$e^{-\tau k}|D_\zeta[g(x-2iK)]| \leq |D_\zeta[f(x)]|$$

and the proof of the lemma is complete. □

Lemma 4.12 *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $\|f\| = M$, *and suppose* $f(z) \neq 0$ *for* $Im(z) > K$ *where* $K \leq 0$. *Then*

$$e^{\tau K}|D_\zeta[f(x+2iK)]| \leq |D_\zeta[g(x)]|$$

for $x \in \mathbf{R}$ *and* $|\zeta| \geq 1$. *Here* $g(z) = e^{i\tau z}\overline{f(\overline{z})}$.

Proof. Note that $g(z)$ has all its zeros in $Im(z) \geq -K \geq 0$. Also, by the arguments used in (4.4) we see that $h_g\left(\frac{-\pi}{2}\right) = \tau$. So, applying Lemma 4.11 to $g(z)$, we get

$$e^{-\tau(-K)}|D_\zeta[f(x-2i(-K))]| \leq |D_\zeta[g(x)]|$$

which gives

$$e^{\tau K}|D_\zeta[f(x+2iK)]| \leq |D_\zeta[g(x)]|.$$

□

The following result is due to Govil and Rahman [25]:

Lemma 4.13 *Let $f(z)$ be an entire function of order 1 and type τ such that $h_f\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = M$, and suppose $f(z)$ has all its zeros in $Im(z) \geq K$ where $K \leq 0$. Then*

$$|f(x+2iK)| \geq e^{(\tau+c)|K|}|f(x)| \text{ for } x \in \mathbf{R}.$$

Lemma 4.14 *Let f and $D_\zeta[f]$ be entire functions of order 1 and type τ such that $h_f\left(\frac{\pi}{2}\right) = 0$, $h_{D_\zeta[f]}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = M$, and suppose $f(z)$ has all its zeros on $Im(z) = K \leq 0$. Then*

$$e^{c|K|}|D_\zeta[f(x)]| \leq |D_\zeta[g(x)]| \text{ for } x \in \mathbf{R}$$

where $|\zeta| \geq 1$ and $g(z) = e^{i\tau z}\overline{f(\overline{z})}$.

Proof. Applying Lemma 4.12 to $f(z)$ we get

$$|D_\zeta[g(x)]| \geq e^{\tau K}|D_\zeta[f(x+2iK)]|. \tag{4.5}$$

By Lemma 4.4, $h_f\left(\frac{-\pi}{2}\right) = \tau$. So $f(x+2iK)$ satisfies the hypotheses of Lemma 4.1 and hence by applying Lemma 4.1, $D_\zeta[f(x+2iK)]|$ has no zeros in $Im(z) < 0$. So $D_\zeta[f(z)]$ has all its zeros in $Im(z) \geq K$. Also, by Theorem 4.8, $D_\zeta[f(x)]$, where $x \in \mathbf{R}$, is bounded by $\frac{\tau}{2}(|\zeta|+1)$. So by Lemma 4.13 applied to $D_\zeta[f(x)]$,

$$|D_\zeta[f(x+2iK)]| \geq e^{(\tau+c)|K|}|D_\zeta[f(x)]| \tag{4.6}$$

for $x \in \mathbf{R}$. Combining (4.5) and (4.6) we get

$$|D_\zeta[g(x)]| \geq e^{c|K|}|D_\zeta[f(x)]|$$

for $x \in \mathbf{R}$. This is equivalent to the conclusion of the lemma. $\square$

Next we need:

Lemma 4.15 *Let f and $D_\zeta[f]$ be entire functions of order 1 and type τ such that $h_f\left(\frac{\pi}{2}\right) = 0$, $h_{D_\zeta[f]}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = M$, $h_{D_\zeta[g]}\left(\frac{\pi}{2}\right) \leq -c < 0$ where $g(z) = e^{i\tau z}\overline{f(\overline{z})}$. Also, suppose $f(z)$ has all its zeros in $Im(z) \leq K \leq 0$. Then*

$$e^{c|K|}|D_\zeta[f(x)]| \leq |D_\zeta[g(x)]|$$

where $|\zeta| \geq 1$.

We omit the proof as it follows on the same lines as Lemma 11 of Govil and Rahman [25].

We can now extend Theorem 4.9 to polar derivatives. In this direction we prove:

Theorem 4.11 *Let $f(z) \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $h_{D_\zeta[f]}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = 1$, and suppose $f(z)$ has all its zeros on $Im(z) = K \leq 0$. Also suppose that the order and type of f and $D_\zeta[f]$ are the same. Then*

$$\|D_\zeta[f]\| \leq \frac{\tau(|\zeta| + 1)}{e^{c|K|} + 1}$$

where $|\zeta| \geq 1$.

Proof. First, suppose that f and $D_\zeta[f]$ are both of order 1 and type τ. Then combining Lemma 4.9 with $Im(z) = y = 0$ and Lemma 4.14, the theorem follows. Now, if f is of type less than τ the result holds trivially. □

If we let $|\zeta| \to \infty$ then Theorem 4.11 reduces to Theorem 4.9. We now extend Theorem 4.10 to polar derivatives.

Theorem 4.12 *Let* $f(z) \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $h_{D_\zeta[f]}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = 1$, $h_{D_\zeta[g]}\left(\frac{\pi}{2}\right) \leq -c < 0$ *where* $g(z) = e^{i\tau z}\overline{f(\overline{z})}$. *Also, suppose* $f(z)$ *has all its zeros in* $Im(z) \leq K \leq 0$. *Then*

$$\|D_\zeta[f]\| \leq \frac{\tau(|\zeta| + 1)}{e^{c|K|} + 1}$$

where $|\zeta| \geq 1$.

Proof. First suppose that f is of order 1 and type τ. Then combining Lemma 4.9 with $Im(z) = y = 0$ and Lemma 4.15, the theorem follows. Now, if f is of type less than τ, the result again holds trivially. □

If we let $|\zeta| \to \infty$, clearly Theorem 4.12 reduces to Theorem 4.10.

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