Eneström-Kakeya Theorem and Some of Its Generalizations

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Abstract. The study of the zeros of polynomials has a very rich history. In addition to having numerous applications, this study has been the inspiration for much theoretical research (including being the initial motivation for modern algebra). The earliest contributors to this subject were Gauss and Cauchy.

Algebraic and analytic methods for finding zeros of a polynomial, in general, can be quite complicated, so it is desirable to put some restrictions on polynomials. Eneström-Kakeya Theorem which is a result in this direction, states that if $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|z| \leq 1$.

Eneström-Kakeya Theorem has been the starting point for considerable literature in Mathematics, concerning the location of the zeros of polynomials. In this article we begin with the earliest results of Eneström and Kakeya and conclude this by presenting some of the recent results on this subject. Our article is expository in nature

1 Introduction and History

The study of the zeros of polynomials has a very rich history. In addition to having numerous applications, this study has been the inspiration for much theoretical research (including being the initial motivation for modern algebra). Algebraic and analytic methods for finding zeros of a polynomial, in general, can be quite complicated, so it is desirable to put some restrictions on polynomials. Historically speaking, the subject dates from about the time when the geometric representation of the complex numbers was introduced into mathematics, and

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the first contributors to the subject were Gauss and Cauchy. Gauss who, as part of his 1816 explorations of the Fundamental Theorem of Algebra, proved (see, for example, [26]):

Theorem 1.1. *If $p(z) = z^n + a_1 z^{n-1} + \cdot + a_n$ is a polynomial of degree n with real coefficients, then all the zeros of p lie in*

$$|z| \leq R = \max_{1\leq k\leq n} \{(n\sqrt{2}|a_k|)^{1/k}\}.$$

In the case of arbitrary real or complex a_j he [26] showed in 1849 that R may be taken as the positive root of the equation

$$z^n - \sqrt{2}(|a_1|z^{n-1} + \cdot + |a_n|) = 0.$$

Cauchy [13, 51] improved the result of Gauss in Theorem 1.1, and proved:

Theorem 1.2. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients, then all the zeros of p lie in $|z| \leq 1 + \max_{0\leq j\leq n-1} \left|\frac{a_j}{a_n}\right|$.*

Notice that neither Theorem 1.1 nor Theorem 1.2 put any restrictions on the coefficients of p (beyond the restriction that they either lie in $\mathbb{R}$ or $\mathbb{C}$, respectively). See [1, 3, 19] for several related results which apply to all polynomials with complex coefficients.

In this survey, we explore the Eneström-Kakeya Theorem and its generalizations. By this, we mean that we explore results which give the location of zeros of a polynomial in terms of their moduli based on hypotheses imposed on the coefficients of the polynomial. We give a mostly chronological presentation. The well-known Eneström-Kakeya Theorem is most commonly stated as follows:

Theorem 1.3. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|z| \leq 1$.*

Proof. Define f by the equation

$$\begin{aligned} p(z)(1-z) &= a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1} \\ &= f(z) - a_n z^{n+1}. \end{aligned}$$

Then for $|z| = 1$, we have

$$\begin{aligned} |f(z)| &\leq |a_0| + |a_1 - a_0| + |a_2 - a_1| + \cdots + |a_n - a_{n-1}| \\ &= a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) \\ &= a_n. \end{aligned}$$

Notice that the function $z^n f(1/z) = \sum_{j=0}^{n}(a_j - a_{j-1})z^{n-j}$, $a_{-1} = 0$ has the same bound on $|z| = 1$ as f. Namely, $|z^n f(1/z)| \leq a_n$ for $|z| = 1$. Since

$z^n f(1/z)$ is analytic in $|z| \le 1$, we have $|z^n f(1/z)| \le a_n$ for $|z| \le 1$ by the Maximum Modulus Theorem. Hence, $|f(1/z)| \le a_n/|z|^n$ for $|z| \le 1$. Replacing z with $1/z$, we see that $|f(z)| \le a_n|z|^n$ for $|z| \ge 1$, and making use of this we get,

$$\begin{aligned} |(1-z)p(z)| &= |f(z) - a_n z^{n+1}| \\ &\ge a_n|z|^{n+1} - |f(z)| \\ &\ge a_n|z|^{n+1} - a_n|z|^n \\ &= a_n|z|^n(|z|-1). \end{aligned}$$

So if $|z| > 1$ then $(1-z)p(z) \ne 0$. Therefore, all the zeros of p lie in $|z| \le 1$. □

The proof given here is modeled on a proof of a generalization of the Eneström-Kakeya Theorem given by Joyal, Labelle, and Rahman [46]. The original statement of the result is slightly different, and has a complicated history.

It seems that G. Eneström was the first to get a result of this nature when he was studying a problem in the theory of pension funds. He published his work in Swedish in 1893 in the journal *Öfversigt af Vetenskaps-Akademiens Förhandlingar* [22]. He mentioned his result again in publications of 1893–94 and 1895. In 1912, S. Kakeya [47] published a paper (in English) in the *Tôhoku Mathematical Journal* which contained the more general result :

Theorem 1.4. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real and positive coefficients, then all the zeros of p lie in the annulus $R_1 \le |z| \le R_2$ where $R_1 = \min_{0 \le j \le n-1} a_j/a_{j+1}$ and $R_2 = \max_{0 \le j \le n-1} a_j/a_{j+1}$.*

In the final few lines of Kakeya's paper, he mentioned that the monotonicity assumption of Theorem 1.3 implies that all zeros of p lie in $|z| \le 1$. The paper gave no references and Kakeya seems to have been unaware of Eneström's earlier work. Kakeya's paper received a bit of attention and was mentioned in at least three other papers in the *Tôhoku Mathematical Journal* during 1912 and 1913; one is in German [42] and two are in English [40, 41]. The two papers in English are by T. Hayashi. At some point, Hayashi must have learned of Eneström's earlier result. Hayashi encouraged Eneström to publish his own results in the *Tôhoku Mathematical Journal* and in 1920, Eneström [23] published in French: "Remarque sur un théorème relatif aux racines de l'equation $a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0 = 0$ où tous les coefficientes a sont réels et positifs" ("Remark on a Theorem on the Roots of the Equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ where all Coefficients are Real and Positive") . In this work [23], Eneström presented a "verbatim" (*textuellement*) translation of his original 1893 paper. We can now see that Eneström was the first to publish a proof of Theorem 1.3 in 1893 and that Kakeya independently proved the result in 1912, this could therefore be a reason to refer to Theorem 1.3 as the "Eneström-Kakeya Theorem." Since Eneström's argument is so historically important, we present a complete English translation of this paper of Eneström [23] in the appendix of this chapter.

2 Generalizations of Eneström-Kakeya Theorem during 1960s

The Eneström-Kakeya Theorem gives an upper bound on the modulus of the zeros of polynomials in a certain class (namely, those polynomials with real, non-negative, monotone increasing coefficients). We can easily obtain a zero-free region for a related class of polynomials in the sense that we can get a lower bound on the modulus of the zeros. By applying Theorem 1.3 to $z^n p(1/z)$ where p has real, non-negative, monotone decreasing coefficients, we get the following:

Theorem 2.1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $a_0 \geq a_1 \geq \cdots \geq a_n \geq 0$, then all the zeros of p lie in $|z| \geq 1$.*

In 1963, Cargo and Shisha [12] introduced the "backward-difference operator" on the coefficients of polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ by defining $\nabla a_j = a_j - a_{j-1}$ (when speaking of ∇a_0 or ∇a_{n+1}, we will assume $a_{-1} = a_{n+1} = 0$). More generally, they also defined "fractional order differences" for any complex α as

$$\nabla^{\alpha} a_n = \sum_{m=0}^{k} (-1)^m \binom{\alpha}{m} a_{k-m}.$$

Cargo and Shisha [12] (see also, [52, 54])

Theorem 2.2. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real, non-negative coefficients satisfying $\nabla^{\alpha} a_j \leq 0$ for $j = 1, 2, \ldots, n$ and $0 < \alpha \leq 1$, then all the zeros of p lie in $|z| \geq 1$.*

Cargo and Shisha showed that Theorem 2.2 reduces to Theorem 2.1 when $\alpha = 1$. They also gave specific polynomials to which Theorem 2.2 applies, but Theorem 2.1 does not, thus showing that the hypotheses are weaker in their result, even though the conclusion is the same as that of Theorem 2.1.

The generalization of Eneström-Kakeya Theorem for functions of several variables was given by Mond and Shisha [53]

In 1967, Joyal, Labelle, and Rahman [46] published a result which might be considered the foundation of the studies which we are currently surveying. The Eneström-Kakeya Theorem, as stated in Theorem 1.3, deals with polynomials with non-negative coefficients which form a monotone sequence. Joyal, Labelle, and Rahman generalized Theorem 1.3 by dropping the condition of non-negativity and maintaining the condition of monotonicity. Namely, they proved:

Theorem 2.3. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|z| \leq (a_n - a_0 + |a_0|)/|a_n|$.*

Of course, when $a_0 \geq 0$ then Theorem 2.3 reduces to Theorem 1.3. The Joyal-Labelle-Rahman result, like the original Eneström-Kakeya Theorem, is only applicable to polynomials with real coefficients. In 1968, Govil and Rahman [30] presented a result that is applicable to polynomials with complex coefficients:

Theorem 2.4. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients satisfying $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots, n$ and $|a_0| \leq |a_1| \leq \cdots \leq |a_n|$, then all the zeros of p lie in $|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|$.*

With $\alpha = \beta = 0$, Theorem 2.4 reduces to Theorem 1.3. In the same paper, Govil and Rahman gave a result for polynomials with complex coefficients but impose a non-negativity and monotonicity condition on the real or imaginary parts of the coefficients of the polynomial:

Theorem 2.5. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying $0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n$, $\alpha_n \neq 0$, then all the zeros of p lie in $|z| \leq 1 + \frac{2}{\alpha_n} \sum_{j=0}^{n} |\beta_j|$.*

With each $\beta_k = 0$, Theorem 2.5 reduces to Theorem 1.3.

3 Generalizations of Eneström-Kakeya Theorem during the 1970s and 1980s

In 1973, Govil and Jain [28] refined Theorem 2.4 by giving a zero-free region about the origin and thus restricting the location of the zeros to an annulus:

Theorem 3.1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients satisfying $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots, n$ and $0 \neq |a_0| \leq |a_1| \leq \cdots \leq |a_n|$, then all the zeros of p lie in*

$$\frac{1}{R^{n-1}[2R(|a_n|/|a_0|) - (\cos\alpha + \sin\alpha)]} \leq |z| \leq R$$

where $R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|$.

In the same paper, Govil and Jain similarly refined Theorem 2.5 by giving a zero-free annular region and improving the outer radius when the real or imaginary part of the coefficients satisfy monotonicity condition:

Theorem 3.2. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying $0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n$, $\alpha_n \neq 0$, then all the zeros of p lie in*

$$\frac{|a_0|}{R^{n-1}[2R\alpha_n + R|\beta_n| - (\alpha_0 + |\beta_0|)]} \leq |z| \leq R$$

where $R = 1 + \frac{1}{\alpha_n}\left(2\sum_{j=0}^{n-1}|\beta_j| + |\beta_n|\right)$.

In a "sequel" paper Govil [29] and Jain further refined Theorems 3.1 and 3.2. The refinement was accomplished by using a more sophisticated technique of proof to improve the inner and outer radii of the annulus containing the zeros of the polynomial. The refinements are, respectively:

Theorem 3.3. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients satisfying* $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ *for some* α *and* β *and for* $j = 0, 1, 2, \ldots, n$ *and* $|a_0| \leq |a_1| \leq \cdots \leq |a_n|$, *then all the zeros of* p *lie in*

$$\frac{1}{2M_2^2}[-R^2|b|(M_2 - |a_0|) + \{4|a_0|R^2M_2^3 + R^4|b|^2(M_2 - |a_0|)^2\}^{1/2}] \leq |z| \leq R$$

where

$$R = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2}$$

and $M_1 = |a_n|r$, $M_2 = |a_n|R^n\left[r + R - \frac{|a_0|}{|a_n|}(\cos\alpha + \sin\alpha)\right]$, $c = |a_n - a_{n-1}|$, $b = a_1 - a_0$, *and* $r = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|$.

Theorem 3.4. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients where* $\mathrm{Re}\, a_j = \alpha_j$ *and* $\mathrm{Im}\, a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$, *satisfying* $0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n$, $\alpha_n \neq 0$, *then all the zeros of* p *lie in*

$$\frac{1}{2M_4^2}\left[-R^2|b|(M_4 - |a_0|) + \{4|a_0|R^2M_4^3 + R^4|b|^2(M_4 - |a_0|)^2\}^{1/2}\right] \leq |z| \leq R$$

where

$$R = \frac{c}{2}\left(\frac{1}{\alpha_n} - \frac{1}{M_3}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{\alpha_n} - \frac{1}{M_3}\right)^2 + \frac{M_3}{\alpha_n}\right\}^{1/2}$$

and $M_3 = \alpha_n r$, $M_4 = R^n\left[(\alpha_n + |\beta_n|)R + \alpha_n r - (\alpha_0 + |\beta_0|)\right]$, $c = |a_n - a_{n-1}|$, $b = a_1 - a_0$, *and* $r = 1 + \frac{1}{\alpha_n}\left(2\sum_{j=0}^{n-1}|\beta_j| + |\beta_n|\right)$.

In 1984, Dewan and Govil [21] considered polynomials with real monotone coefficients and obtained:

Theorem 3.5. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with real coefficients satisfying* $a_0 \leq a_1 \leq \cdots \leq a_n$, *then all the zeros of* p *lie in*

$$\frac{1}{2M_2^2}\left[-R^2b(M_2 - |a_0|) + \{4|a_0|R^2M_2^3 + R^4b^2(M_2 - |a_0|)^2\}^{1/2}\right] \leq |z| \leq R$$

where

$$R = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2}$$

and $M_1 = a_n - a_0 + |a_0|$, $M_2 = R^n(|a_n|R + a_n - a_0)$, $c = a_n - a_{n-1}$, *and* $b = a_1 - a_0$.

Dewan and Govil also showed that $R \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$ and that the inner radius of the zero containing region is less than 1, indicating that this result is an improvement of the result of Joyal, Labelle, and Rahman (Theorem 2.3); hence it is also an improvement of the Eneström-Kakeya Theorem. They also gave specific examples of polynomials for which their result gives sharper bound than obtainable from Theorem 2.3 of Joyal, Labelle and Rahman.

In 1980, Aziz and Mohammad [6] introduced a condition on the coefficients to produce the following generalization of Theorem 1.3:

Theorem 3.6. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with real, positive coefficients. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2} \geq 0, \text{ for } j = 1, 2, \ldots, n+1$$

where we take $a_{-1} = a_{n+1} = 0$, *then all zeros of* p *lie in* $|z| \leq t_1$.

With $t_1 = 1$ and $t_2 = 0$, Theorem 3.6 implies the Eneström-Kakeya Theorem. In the same paper, Aziz and Mohammad [6] introduced an interesting and general condition on the coefficients of a power series representation $\sum_{j=0}^{\infty} a_j z^j$ of an analytic function in order to restrict the location of the zeros. The condition is that $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots$ and $|a_0| \leq t|a_1| \leq \cdots \leq t^{k-1}|a_{k-1}| \leq t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots$ for some $t > 0$ and some $k = 0, 1, \ldots$. Aziz and Mohammad [7] imposed similar conditions on the coefficients of polynomials and proved the following three theorems.

Theorem 3.7. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients satisfying*

$$|a_0| \leq t|a_1| \leq \cdots \leq t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^n|a_n|$$

for some $k = 0, 1, \ldots, n$ *and some* $t > 0$, *then all zeros of* p *lie in*

$$|z| \leq t\left(\frac{2t^k|a_k|}{t^n|a_n|} - 1\right) + 2\sum_{j=0}^{n} \frac{|a_j - |a_j||}{t^{n-j-1}|a_n|}.$$

Theorem 3.8. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients where* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$, *satisfying*

$$0 \leq \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^n\alpha_n > 0$$

for some $k = 0, 1, \ldots, n$ *and some* $t > 0$, *then all zeros of* p *lie in*

$$|z| \leq t\left(\frac{2t^k\alpha_k}{t^n\alpha_n} - 1\right) + \frac{2}{\alpha_n}\sum_{j=0}^{n} \frac{|\beta_j|}{t^{n-j-1}}.$$

Theorem 3.9. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$0 \leq \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^n \alpha_n > 0$$

and

$$0 \leq \beta_0 \leq t\beta_1 \leq \cdots \leq t^r \beta_r \geq t^{r+1}\alpha_{r+1} \geq \cdots \geq t^n \beta_n \geq 0$$

for some $k = 0, 1, \ldots, n$, some $r = 0, 1, \ldots, n$, and some $t > 0$, then all zeros of p lie in

$$|z| \leq \frac{t}{|a_n|}\{2(t^{k-n}\alpha_k + t^{r-n}\beta_r) - (\alpha_n + \beta_n)\}.$$

Notice that each of the three previous results imply Theorem 1.3 for the appropriate choices of t, k, and β_j.

4 Generalizations of Eneström-Kakeya Theorem during 1990s

In the style of Aziz and Mohammad [7], Dewan and Bidkham [20] dropped the non-negativity condition of Theorem 3.8 and proved for polynomials with real coefficients:

Theorem 4.1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying*

$$a_0 \leq ta_1 \leq \cdots \leq t^k a_k \geq t^{k+1} a_{k+1} \geq \cdots \geq t^n a_n$$

for some $k = 0, 1, \ldots, n$ and some $t > 0$, then all zeros of p lie in

$$|z| \leq t\left(\frac{2t^k a_k}{t^n |a_n|} - \frac{a_n}{|a_n|}\right) + \frac{1}{t^{n-1}|a_n|}(|a_0| - a_0).$$

With $a_0 > 0$ and $a_n > 0$ in Theorem 4.1, we see that the zeros of p lie in

$$|z| \leq t\left(\frac{2t^k a_k}{t^n a_n} - 1\right).$$

The above result also follows from Theorem 3.8 if we take each $\beta_j = 0$, and in this sense Dewan and Bidkham's result overlaps with that of Aziz and Mohammad [7].

Related to the hypotheses of Theorem 4.1, Gardner and Govil [24] proved the following in 1994 which was inspired by a result by Aziz and Mohammad [6] for analytic functions:

Theorem 4.2. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_0 \le t\alpha_1 \le \cdots \le t^k \alpha_k \ge t^{k+1}\alpha_{k+1} \ge \cdots \ge t^n \alpha_n$$

and

$$\beta_0 \le t\beta_1 \le \cdots \le t^r \beta_r \ge t^{r+1}\beta_{r+1} \ge \cdots \ge t^n \beta_n$$

for some $k = 0, 1, \ldots, n$, some $r = 0, 1, \ldots, n$, and some $t > 0$, then all zeros of p lie in $R_1 \le |z| \le R_2$, where

$$R_1 = \min\left\{(t|a_0|/(2(t^k\alpha_k + t^r\beta_r) - (\alpha_0 + \beta_0) - t^n(\alpha_n + \beta_n - |a_n|)), t\right\}$$

and

$$R_2 = \max\left\{\left[|a_0|t^{n+1} - t^{n-1}(\alpha_0 + \beta_0) - t(\alpha_n + \beta_n) + (t^2+1)(t^{n-k-1}\alpha_k \right.\right.$$

$$+t^{n-r-1}\beta_r) + (t^2 - 1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right)$$

$$\left.\left. +(1-t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right)\right] \Big/ |a_n|, \frac{1}{t}\right\}.$$

The flexibility of Theorem 4.2 is revealed by considering the corollaries which result by letting $t = 1$, and $k, r \in \{0, n\}$. For example, with $t = 1$, $k = n$ and $r = n$, it implies the following which is clearly a generalization and refinement of the result of Joyal, Labelle, and Rahman (Theorem 2.3):

Corollary 4.3. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \text{ and } \beta_0 \le \beta_1 \le \cdots \le \beta_n$$

for some $t > 0$, then all zeros of p lie in

$$\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \le |z| \le \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.$$

By making suitable choice of t and k and using appropriate transformations Gardner and Govil [24] also obtained several results analogous to the above corollary when, for example, real parts of the coefficients is monotonically decreasing and imaginary parts monotonically decreasing, or real parts of the coefficients monotonic increasing and imaginary parts monotonic decreasing.

In order to apply the above Theorem 4.2 of Gardner and Govil [24] both the real and imaginary parts of the coefficients have to be monotonic but if this does not happen and instead only the real or imaginary parts of the coefficients satisfy this condition then the Theorem 4.2 is not applicable. In this regard, Gardner and Govil [25] proved a result related to Theorem 4.2, but with hypotheses restricted to just the the real parts or imaginary parts of the coefficients. To be more precise their result is the following:

Theorem 4.4. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where Re $a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^n \alpha_n$$

for some $k = 0, 1, \ldots, n$, and some $t > 0$, then all zeros of p lie in $R_1 \leq |z| \leq R_2$, where

$$R_1 = t|a_0| \Bigg/ \left(2(t^k\alpha_k - \alpha_0 - t^n\alpha_n + t^n|a_n| + |\beta_0| + |\beta_n|t^n + 2\sum_{j=1}^{n-1} |\beta_j| t^j \right)$$

and

$$R_2 = \max \left\{ (|a_0|t^{n+1} + (t^2+1)t^{n-k-1}\alpha_k - t^{n-1}\alpha_0 - t\alpha_n + (t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j \right.$$

$$\left. +(1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j \ + \sum_{j=1}^{n}(|\beta_{j-1}| + t|\beta_j|)t^{n-j} \right) \Bigg/ |a_n|, \frac{1}{t} \Bigg\}.$$

By using suitable transformations, Gardner and Govil [25] also obtained results analogous to the above Theorem 4.4 when the condition is satisfied by imaginary parts of the coefficients.

In the same paper that contained Theorem 4.1, Dewan and Bidkham [20] also generalized Theorem 3.5 due to Dewan and Govil, and proved the following:

Theorem 4.5. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying*

$$a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$$

for some $k = 0, 1, \ldots, n$, then all zeros of p lie in

$$\frac{1}{2M_2^2}\left[-R^2 b(M_2 - |a_0|) + \{4|a_0|R^2M_2^3 + R^4b^2(M_2 - |a_0|)^2\}^{1/2}\right] \leq |z| \leq R$$

where

$$R = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2}$$

and $M_1 = -a_n + 2a_k - a_0 + |a_0|$, $M_2 = R^n(|a_n|R + 2a_k - a_n - a_0)$, $c = |a_n - a_{n-1}|$, and $b = a_1 - a_0$.

In 1998, Aziz and Shah [8] introduced a very general condition on the coefficients of a polynomial. Though the condition is complicated, it allowed them to conclude several of the previous results mentioned above. They proved:

Theorem 4.6. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. Suppose for some $t > 0$ we have*

$$\max_{|z|=r} |ta_0 z^{n+1} + (ta_1 - a_0)z^n + \cdots + (ta_n - a_{n-1})z| \leq M_1$$

and

$$\max_{|z|=r} | - a_n z^{n+1} + (ta_n - a_{n-1})z^n + \cdots + (ta_1 - a_0)z| \leq M_2,$$

where r is any positive real number. Then all zeros of p lie in

$$\frac{1}{2M_2^2}\left[-r^2 b(M_2 - t|a_0|) + \{4t|a_0|r^2 M_2^3 + r^4 b^2 (M_2 - t|a_0|)^2\}^{1/2}\right] \leq |z| \leq R$$

where

$$R = 2M_1^2[-c(M_1 - |a_n|)r^2 + \{4|a_n|r^2 M_1^3 + r^4 c^2 (M_1 - |a_n|)^2\}^{1/2}]^{-1},$$

$c = |ta_n - a_{n-1}|$, *and* $b = |ta_1 - a_0|$.

Aziz and Shah proved that Theorem 4.6 implies Theorem 3.3 due to Govil and Jain and stated that a similar argument shows that Theorem 4.6 implies Dewan and Bidkham's Theorem 4.5, as well as Govil and Jain's Theorem 3.4. In the same paper, Aziz and Shah also gave the following result with similar type of hypotheses which implies Theorem 4.1 due to Dewan and Bidkham:

Theorem 4.7. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. Suppose for some $t > 0$ we have*

$$\max_{|z|=r} |ta_0 z^n + (ta_1 - a_0)z^{n-1} + \cdots + (ta_n - a_{n-1})| \leq M$$

where r is any positive real number. Then all zeros of p lie in

$$|z| \leq \max\left\{\frac{M}{|a_n|}, \frac{1}{r}\right\}.$$

Aziz and Zargar [9] relaxed the monotonicity condition of Joyal, Labelle, and Rahman [46] and obtained a result related to the Eneström-Kakeya Theorem. Here,the disk obtained is not necessarily centered at the origin. This result involves a modification of the monotonicity condition by introducing a parameter λ, in the sense that the first $(n - 1)$ coefficients of the polynomial satisfy the monotonicity condition while the last coefficient a_n does not follow this pattern, and is free. This λ condition will appear often in research on the Eneström-Kakeya Theorem in the new millennium:

Theorem 4.8. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients satisfying $a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq \lambda a_n$. Then all the zeros of p lie in $|z + (\lambda - 1)| \leq (\lambda a_n - a_0 + |a_0|)/|a_n|$.*

Of course, with $\lambda = 1$, Theorem 4.8 reduces to Joyal, Labelle, and Rahman's Theorem 2.3. With $\lambda = a_{n-1}/a_n$, we see from Theorem 4.8 that the zeros of a polynomial with monotone coefficients $a_0 \leq a_1 \leq \cdots \leq a_{n-1}$ has all its zeros in $|z + (a_{n-1}/a_n - 1)| \leq (a_{n-1} - a_0 + |a_0|)/|a_n|$. Later, Aziz and Zargar [9] generalized their own Theorem 4.8 and proved :

Theorem 4.9. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying*

$$a_0 \leq ta_1 \leq \cdots \leq t^k a_k \geq t^{k+1} a_{k+1} \geq \cdots \geq t^n a_n$$

for some $k = 0, 1, \ldots, n-1$ and some $t > 0$, then all zeros of p lie in

$$\left| z + \left(\frac{a_{n-1}}{a_n} - t \right) \right| \leq t \left(\frac{2t^k a_k}{t^n |a_n|} - \frac{a_{n-1}}{t|a_n|} \right) + \frac{1}{t^{n-1}|a_n|}(|a_0| - a_0).$$

In the same paper, Aziz and Zargar [9] proved a result related to Theorem 4.9, but with a hypothesis concerning the even indexed and odd indexed coefficients separately. Their result is:

Theorem 4.10. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients satisfying*

$$0 < a_0 \leq t^2 a_2 \leq t^4 a_4 \leq \cdots \leq t^{2\lfloor n/2 \rfloor} a_{2\lfloor n/2 \rfloor}$$

and

$$0 < a_1 \leq t^2 a_3 \leq t^4 a_5 \leq \cdots \leq t^{2\lfloor n/2 \rfloor} a_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then all of the zeros of p lie in

$$\left| z - \frac{a_{n-1}}{a_n} \right| \leq t + \frac{a_{n-1}}{a_n}.$$

5 Generalizations in the New Millennium

Cao and Gardner [11] extended the even and odd indexed coefficient condition of Aziz and Zargar's Theorem 4.10 to polynomials with complex coefficients to prove the following :

Theorem 5.1. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_0 \leq t^2 \alpha_2 \leq t^4 \alpha_4 \leq \cdots \leq t^{2k} \alpha_{2k} \geq t^{2k+2} \alpha_{2k+2} \geq \cdots \geq t^{2\lfloor n/2 \rfloor} \alpha_{2\lfloor n/2 \rfloor},$$

$$\alpha_1 \le t^2\alpha_3 \le t^4\alpha_5 \le \cdots \le t^{2\ell-2}\alpha_{2\ell-1} \ge t^{2\ell}\alpha_{2\ell+1} \ge \cdots \ge t^{2\lfloor n/2\rfloor}\alpha_{2\lfloor (n+1)/2\rfloor-1},$$

$$\beta_0 \le t^2\beta_2 \le t^4\beta_4 \le \cdots \le t^{2s}\beta_{2s} \ge t^{2s+2}\beta_{2s+2} \ge \cdots \ge t^{2\lfloor n/2\rfloor}\beta_{2\lfloor n/2\rfloor},$$

and

$$\beta_1 \le t^2\beta_3 \le t^4\beta_5 \le \cdots \le t^{2q-2}\beta_{2q-1} \ge t^{2q}\beta_{2q+1} \ge \cdots \ge t^{2\lfloor n/2\rfloor}\beta_{2\lfloor (n+1)/2\rfloor-1}$$

for some k, ℓ, s, q in $\{0, 1, \ldots, \lfloor n/2\rfloor\}$. Then all the zeros of p lie in $R_1 \le |z| \le R_2$ where $R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\}$, $R_2 = \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\}$ and

$$M_1 = -(\alpha_0+\beta_0) + (|\alpha_1|+|\beta_1|)t - (\alpha_1+\beta_1)t$$
$$+2[\alpha_{2k}t^{2k} + 2_{2\ell-1}t^{2\ell-1} + \beta_{2s}t^{2s} + \beta_{2q-1}t^{2q-1}] - (\alpha_{n-1}+\beta_{n-1})t^{n-1}$$
$$-(\alpha_n+\beta_n)t^n + (|\alpha_{n-1}|+|\beta_{n-1}|)t^{n-1} + (|\alpha_n|+|\beta_n|)t^n$$

$$M_2 = t^{n+3}(|a_0|-\alpha_0-\beta_0)+(|\alpha_1|-\alpha_1-\beta_1)t^{n+2}+(t^4+1)(\alpha_{2k}t^{n-1-2k}+\alpha_{2\ell-1}t^{n-2\ell}$$
$$+\beta_{2s}t^{n-1-2s} + \beta_{2q-1}t^{n-2q}) - (\alpha_{n-1}+\beta_{n-1}) + |a_{n-1}| - (\alpha_n+\beta_n)t^{-1}$$
$$+(t^4-1)\left(\sum_{j=0,\, j\ \text{even}}^{2k-2}\alpha_j t^{n-1-j} + \sum_{j=1,\, j\ \text{odd}}^{2\ell-3}\alpha_j t^{n-1-j} + \sum_{j=0,\, j\ \text{even}}^{2s-2}\beta_j t^{n-1-j}\right.$$
$$+\sum_{j=1,\, j\ \text{odd}}^{2q-3}\beta_j t^{n-1-j} - \sum_{j=2k+2,\, j\ \text{even}}^{2\lfloor n/2\rfloor}\alpha_j t^{n-1-j} - \sum_{j=2\ell+1,\, j\ \text{odd}}^{2\lfloor (n+1)/2\rfloor-1}\alpha_j t^{n-1-j}$$
$$\left.-\sum_{j=2s+2,\, j\ \text{even}}^{2\lfloor n/2\rfloor}\beta_j t^{n-1-j} - \sum_{j=2q+1,\, j\ \text{odd}}^{2\lfloor (n+1)/2\rfloor-1}\beta_j t^{n-1-j}\right).$$

The flexible hypotheses of Theorem 5.1 allow to obtain a large number of corollaries. For example, monotonicity conditions can be imposed on the even and odd indexed coefficients to prove the following result for polynomials with real coefficients:

Corollary 5.2. *Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with real coefficients satisfying*

$$a_0 \le a_2 \le a_4 \le \cdots a_{2\lfloor n/2\rfloor},$$

and

$$a_1 \ge a_3 \ge a_5 \ge \cdots a_{2\lfloor (n+1)/2\rfloor-1}.$$

Then all the zeros of p lie in $R_1 \le |z| \le R_2$ where $R_1 = \min\left\{\frac{|a_0|}{M_1}, 1\right\}$ and $R_2 = \max\left\{\frac{M_2}{|a_n|}, 1\right\}$ for $M_1 = -a_0+|a_1|+a_1+2a_{2\lfloor n/2\rfloor}+|a_{n-1}|-a_{n-1}+|a_n|-a_n$ and $M_2 = |a_0| - a_0 + |a_1| + a_1 + 2a_{2\lfloor n/2\rfloor} + |a_{n-1}| - a_{n-1} - a_n$.

Cao and Gardner gave specific examples of polynomials showing that these results sometimes give improvements over previous results. In addition, they [11] addressed a similar condition on the moduli of the even and odd indexed coefficients for polynomials with complex coefficients:

Theorem 5.3. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying $|\arg a_j - \beta| \le \alpha \le \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots, n$ and*

$$|a_0| \le t^2|a_2| \le t^4|a_4| \le \cdots \le t^{2k}|a_{2k}| \ge t^{2k+2}|a_{2k+2}| \ge \cdots \ge t^{2\lfloor n/2 \rfloor}|a_{2\lfloor n/2 \rfloor}|,$$

$$|a_1| \le t^2|a_3| \le t^4|a_5| \le \cdots \le t^{2\ell-2}|a_{2\ell-1}| \ge t^{2\ell}|a_{2\ell+1}| \ge \\ \cdots \ge t^{2\lfloor n/2 \rfloor}|a_{2\lfloor (n+1)/2 \rfloor -1}|$$

for some $k = 0, 1, \ldots, \lfloor n/2 \rfloor$ and $\ell = 0, 1, \ldots, \lfloor n/2 \rfloor$. Then all the zeros of p lie in $R_1 \le |z| \le R_2$ where $R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\}$, $R_2 = \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\}$,

$$M_1 = |a_1|t + |a_{n-1}|t^{n-1} + |a_n|t^n + \cos\alpha[-|a_0| - |a_1|t + 2|a_{2k}|t^{2k} + 2|a_{2\ell-1}|t^{2\ell-1}$$

$$-|a_{n-1}|t^{n-1} - |a_n|t^n] + \sin\alpha\left[2\sum_{j=0}^{n-2}|a_j|t^j + |a_0| + |a_1|t + |a_{n-1}|t^{n-1} + |a_n|t^n\right]$$

and

$$M_2 = |a_0|t^{n+3} + |a_1|t^{n+2} + |a_{n-1}| + \cos\alpha\left\{(t^4-1)\left(\sum_{j=0,\, j \text{ even}}^{2k-2}|a_j|t^{n-1-j}\right.\right.$$

$$\left.+ \sum_{j=1,\, j \text{ odd}}^{2\ell-3}|a_j|t^{n-1-j} - \sum_{j=2k+2,\, j \text{ even}}^{2\lfloor n/2 \rfloor}|a_j|t^{n-1-j} - \sum_{j=2\ell+1,\, j \text{ odd}}^{2\lfloor (n+1)/2 \rfloor -1}|a_j|t^{n-1-j}\right)$$

$$\left.+(t^4+1)(|a_{2k}|t^{n-1-2k} + |a_{2\ell-1}|t^{n-2\ell}) - |a_0|t^{n+3} - |a_1|t^{n+2} - |a_{n-1}| - |a_n|t^{-1}\right\}$$

$$+\sin\alpha\left\{(t^4+1)\sum_{j=2}^{n-2}|a_j|t^{n-1-j} + |a_0|t^{n-1} + |a_1|t^{n-2} + |a_{n-1}|t^4 + |a_n|t^3\right\}.$$

The hypotheses of Theorem 5.3, as well as several of the other results above, involve a reversal of an inequality condition on the coefficients of a polynomial. In 2005, Chattopadhyay, Das, Jain, and Konwar [14] took this idea of a reversal of the inequality to its logical conclusion, and introduced hypotheses concerning an arbitrary number of reversals in an inequality on the coefficients. As an example, they [14] proved:

Theorem 5.4. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying for some $t > 0$*

$$\alpha_0 \leq t\alpha_1 \leq \cdots \leq t^{k_1}\alpha_{k_1} \geq t^{k_1+1}\alpha_{k_1+1} \geq \cdots \geq t^{k_2}\alpha_{k_2} \leq t^{k_2+1}\alpha_{k_2+1} \leq \cdots t^n\alpha_n$$

and

$$\beta_0 \leq t\beta_1 \leq \cdots \leq t^{r_1}\beta_{r_1} \geq t^{r_1+1}\beta_{r_1+1} \geq \cdots \geq t^{r_2}\beta_{r_2} \leq t^{r_2+1}\beta_{r_2+1} \leq \cdots t^n\beta_n,$$

where the inequalities involving the real parts reverse at each of the indices $k_1, k_2, \ldots, k_p$, and the inequalities involving the imaginary parts reverse at each of the indices $r_1, r_2, \ldots, r_q$. Then all zeros of p lie in $R_1 \leq |z| \leq R_2$, where

$$R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\}, R_2 = \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\},$$

$$M_1 = -\left(\alpha_0 + (-1)^{p+1}\alpha_n t^n + \sum_{j=1}^{p}(-1)^j \alpha_{k_j} t^{k_j}\right)$$

$$-\left(\beta_0 + (-1)^{q+1}\beta_n t^n + \sum_{j=1}^{q}(-1)^j \beta_{r_j} t^{r_j}\right) + |a_n|t^n,$$

and

$$M_2 = \left[-\alpha_0 t^{n-1} + (-1)^{p+1}\alpha_n t + (t^2+1)\sum_{j=1}^{p}(-1)^j \alpha_{k_j} t^{n-k_j-1}\right.$$

$$\left. +(t^2-1)\sum_{j=0}^{p}\left\{(-1)^{j+1}\sum_{m=k_j+1}^{k_{j+1}-1}\alpha_m t^{n-m-1}\right\}\right]$$

$$-\left[\beta_0 t^{n-1} + (-1)^{q+1}\beta_n t + (t^2+1)\sum_{j=1}^{q}(-1)^j \beta_{r_j} t^{n-r_j-1}\right.$$

$$\left. +(t^2-1)\sum_{j=0}^{q}\left\{(-1)^{j+1}\sum_{m=r_j+1}^{r_{j+1}-1}\beta_m t^{n-m-1}\right\}\right] + |a_0|t^{n+1},$$

where we take $k_0 = r_0 = 0$ and $k_{p+1} = r_{q+1} = n$.

With $k_1 = k$, $r_1 = r$, and the number of reversals $p = q = 1$, Theorem 5.4 reduces to Gardner and Govil's Theorem 4.2. In the same paper, Chattopadhyay, Das, Jain, and Konwar also gave a result which hypothesizes a number of reversals in an inequality concerning the moduli of the coefficients, thus giving a generalization of Theorem 3.7 due to Aziz and Mohammad.

In 2007, Shah and Liman [56] extended Aziz and Zargar's idea from Theorem 4.8 to complex polynomials by hypothesizing a condition on the moduli of the polynomial. Their result is as follows:

Theorem 5.5. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ where $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β for $j = 0, 1, 2, \ldots, n$. If for some $\lambda \geq 1$ we have*

$$|a_0| \leq |a_1| \leq \ldots |a_{n-1}| \leq \lambda |a_n|$$

then all the zeros of p lie in

$$|z + (\lambda - 1)| \leq \left\{ (\lambda |a_n| - |a_0|)(\sin\alpha + \cos\alpha) + |a_0| + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \right\} \Big/ |a_n|.$$

In the same paper, Shah and Liman produced similar results by imposing the "λ condition" on the real parts and by combining this with a reversal in the monotonicity condition.

In 2009, in a paper dealing mostly with the number of zeros in a region, Jain [44] produced a corollary involving a fairly simple monotonicity condition very similar to the original Eneström-Kakeya Theorem, but combined with an additional hypothesis on coefficients a_0, a_{n-1}, and a_n:

Theorem 5.6. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $0 < a_0 \leq a_1 \leq \cdots \leq a_{n-1} < a_n$ and such that*

$$(n+1)^n a_n^{n-1} \{(n+1) a_0 a_n + n(a_n - a_{n-1})(a_{n-1} - a_0)\} < n^n (a_n - a_{n-1})^{n+1},$$

then all the zeros of p lie in

$$|z| \leq \frac{n}{n+1} \frac{a_n - a_{n-1}}{a_n} < 1.$$

Jain [44] also showed by example that for some polynomials satisfying both the hypotheses of the Eneström-Kakeya Theorem and the hypotheses of his Theorem 5.6, the location of the zeros can be more finely constrained by his result than by the Eenström-Kakeya Theorem (which will, of course, restrict the zeros to $|z| \leq 1$).

Choo [16] generalized Theorem 5.3 by introducing another parameter in each of the monotonicity-type hypotheses on the coefficients. In addition, he gave a simpler expression for the upper bound on the zero containing region:

Theorem 5.7. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots, n$ and*

$$|a_0| \leq t^2 |a_2| \leq t^4 |a_4| \leq \cdots \leq t^{2k} |a_{2k}| \geq t^{2k+2} |a_{2k+2}| \geq \cdots \geq \lambda_1 t^{2\lfloor n/2 \rfloor} |a_{2\lfloor n/2 \rfloor}|,$$

$$|a_1| \leq t^2 |a_3| \leq t^4 |a_5| \leq \cdots \leq t^{2\ell - 2} |a_{2\ell - 1}| \geq t^{2\ell} |a_{2\ell + 1}| \geq \\ \cdots \geq \lambda_2 t^{2\lfloor n/2 \rfloor} |a_{2\lfloor (n+1)/2 \rfloor - 1}|$$

for some $k = 0, 1, \ldots, \lfloor n/2 \rfloor$, $\ell = 0, 1, \ldots, \lfloor n/2 \rfloor$, $\lambda_1 > 0$, *and* $\lambda_2 > 0$. *Then all the zeros of* p *lie in* $R_1 \leq |z| \leq R_2$ *where* $R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\}$, $R_2 = \max\left\{\frac{M_2}{t^{n-1}|a_n|}, \frac{1}{t}\right\}$,

$$M_1 = |a_1|t + |a_{n-1}|t^{n-1} + |a_n|t^n + |(\lambda^{**} - 1)a_{n-1}|t^{n-1} + |(\lambda^* - 1)a_n|t^n$$

$$+ \cos\alpha[-|a_0| - |a_1|t + 2|a_{2k}|t^{2k} + 2|a_{2\ell-1}|t^{2\ell-1} - \lambda^{**}|a_{n-1}|t^{n-1} - \lambda^*|a_n|t^n]$$

$$+ \sin\alpha\left[2\sum_{j=0}^{n-2}|a_j|t^j + |a_0| + |a_1|t + \lambda^{**}|a_{n-1}|t^{n-1} + \lambda^*|a_n|t^n\right]$$

and

$$M_2 = |a_0| + |a_1|t + |a_{n-1}|t^{n-1} + |(\lambda^{**} - 1)a_{n-1}|t^{n-1} + |(\lambda^* - 1)a_n|t^n$$

$$+ \cos\alpha\left[2|a_{2k}|t^{2k} + 2|a_{2\ell-1}|t^{2\ell-1} - \lambda^*|a_n|t^n - \lambda^{**}|a_{n-1}|t^{n-1} - |a_1|t - |a_0|\right]$$

$$+ \sin\alpha\left\{\lambda^*|a_n|t^n + \lambda^{**}|a_{n-1}|t^{n-1} + |a_1|t + |a_0| + 2\sum_{j=2}^{n-2}|a_j|t^j\right\}.$$

For n *even we have* $\lambda^* = \lambda_1$ *and* $\lambda^{**} = \lambda_2$, *but for* n *odd we have* $\lambda^* = \lambda_2$ *and* $\lambda^{**} = \lambda_1$.

In the same paper, Choo [16] gave a similar generalization of Theorem 5.1 due to Cao and Gardener.

In 2010, Singh and Shah [57] combined the hypotheses of Aziz and Mohammad's Theorem 3.6 (but applied to complex coefficients, as opposed to real coefficients) with the hypotheses of Aziz and Zargar's Theorem 4.8 to get the following:

Theorem 5.8. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients where* $\mathrm{Re}\, a_j = \alpha_j$ *and* $\mathrm{Im}\, a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$. *If* $t_1 > t_2 \geq 0$, *can be found such that we have*

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0 \text{ for } j = 2, 3, \ldots, n,$$

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0 \text{ for } j = 2, 3, \ldots, n,$$

where we take $\alpha_{n+1} = \beta_{n+1} = 0$, *and for some* $\lambda \geq 1$,

$$\lambda\alpha_n(t_1 - t_2) - \alpha_{n-1} \geq 0 \text{ and } \lambda\beta_n(t_1 - t_2) - \beta_{n-1} \geq 0,$$

then all zeros of p *lie in* $|z + (\lambda - 1)(t_1 - t_2)| \leq R$ *where*

$$R = \frac{1}{|\alpha_n|}\left\{\lambda(\alpha_n + \beta_n)(t_1 - t_2) + (\alpha_n + \beta_n)t_2 - (\alpha_1 + \beta_1)\frac{t_2}{t_1^{n-1}} - (\alpha_0 + \beta_0)\frac{1}{t_1^{n-1}}\right.$$

$$\left. + (|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|)\frac{1}{t_1^n} + (|\alpha_0| + |\beta_0|)\frac{t_2}{t_1^n}\right\}.$$

With all the coefficients real and positive, and $\lambda = 1$, the Theorem 5.8 reduces to Theorem 3.6 due to Aziz and Mohammad. With $t_1 = 1$, $t_2 = 0$, and $\lambda = 1$, Theorem 5.8 reduces to Theorem 4.8 of Aziz and Zargar. In the same paper, Singh and Shah [57] modified the hypotheses

$$\lambda\alpha_n(t_1 - t_2) - \alpha_{n-1} \geq 0 \text{ and } \lambda\beta_n(t_1 - t_2) - \beta_{n-1} \geq 0,$$

to

$$\lambda_1\alpha_n(t_1 - t_2) - \alpha_{n-1} \geq 0 \text{ and } \lambda_2\beta_n(t_1 - t_2) - \beta_{n-1} \geq 0$$

where $\lambda_1 \geq 1$ and $\lambda_2 \geq 1$, and proved a result concerning the location of zeros in a disk (not necessarily centered at origin) which includes many of the other results mentioned above. In a related result, but concerning zeros in a disk centered at origin, Singh and Shah [58] in 2011 presented the following:

Theorem 5.9. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$. If $t_1 > t_2 \geq 0$ can be found such that for $j = 1, 2, \ldots, n+1$ we have*

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0$$

and

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0,$$

where we take $\alpha_{-1} = \alpha_{n+1} = \beta_{-1} = \beta_{n+1} = 0$. Then all zeros of p lie in $|z| \leq (|\alpha_n + \beta_n + M|t_1/|a_n|$ where

$$\begin{aligned} M = {} & -\alpha_1 \frac{t_2}{t_1^n} - \frac{\alpha_0}{t_1^n} + |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| \frac{1}{t_1^{n+1}} + |\alpha_0 t_1 t_2| \frac{1}{t_1^{n+2}} \\ & -\beta_1 \frac{t_2}{t_1^n} - \frac{\beta_0}{t_1^n} + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)| \frac{1}{t_1^{n+1}} + |\beta_0 t_1 t_2| \frac{1}{t_1^{n+2}}. \end{aligned}$$

Again, this result implies Aziz and Mohammad's Theorem 3.6.

Using the same hypotheses as Theorem 5.9, Singh and Shah [59] proved another result concerning the location of zeros, but this time obtained an annulus region containing all the zeros:

Theorem 5.10. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$. If $t_1 \geq t_2$, $t_1 \neq 0$, can be found such that for $j = 1, 2, \ldots, n+1$ we have*

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0$$

and

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0,$$

where we take $\alpha_{-1} = \alpha_{n+1} = \beta_{-1} = \beta_{n+1} = 0$. Then all zeros of p lie in $\min\{R_2, 1/t_1\} \leq |z| \leq \max\{R_1, t_1\}$. Here,

$$R_1 = \Bigg\{ -(|a_n| - K_1)|a_n(t_1 - t_2) - a_{n-1}| + \big[(|a_n| - K_1)^2 |a_n(t_1 - t_2) - a_{n-1}|^2$$

$$+4K_1^3t_1^2|a_n|\big]^{1/2}\Bigg\}/(2K_1|a_n|),$$

$$R_2=\Bigg\{-(|a_0|t_1t_2-K_2)|a_1t_1t_2+a_0(t_1-t_2)|t_1^2$$

$$+\left[(|a_0|t_1t_2-K_2)^2|a_1t_1t_2+a_0(t_1-t_2)|^2t_1^4+4K_2^3|a_0|t_1^3t_2\right]^{1/2}\Bigg\}/(2K_2^2),$$

$K_1=(\alpha_n+\beta_n)+(|\alpha_0|-\alpha_0)t_2/t_1^{n+1}+(|\beta_0|-\beta_0)t_2/t_1^{n+1}$, and $K_2=(\alpha_n+\beta_n)t_1^{n+2}+(|\alpha_n|+|\beta_n|)t_1^{n+2}-(\alpha_0+\beta_0)t_1t_2$.

When each $\beta_j=0$, Theorem 5.10 reduces to Theorem 3.6. With all coefficients real and positive, and $t_1=1$ and $t_2=0$, Theorem 5.10 implies the following clean refinement of Theorem 1.3:

Corollary 5.11. *Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with real coefficients satisfying $0\le a_0\le a_1\le\cdots\le a_n$, then all the zeros of p lie in $\dfrac{a_0}{2a_n}\le|z|\le 1$.*

In the same paper, Singh and Shah [59] introduced a reversal in the inequality imposed on the coefficients at a particular point and proved :

Theorem 5.12. *Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients where $Re\,a_j=\alpha_j$ and $Im\,a_j=\beta_j$ for $j=0,1,2,\dots,n$. If $t_1>t_2\ge 0$ can be found such that*

$$\alpha_jt_1t_2+\alpha_{j-1}(t_1-t_2)-\alpha_{j-2}\ge 0,\ \text{for } j=2,3,\dots,r+1,$$

$$\alpha_jt_1t_2+\alpha_{j-1}(t_1-t_2)-\alpha_{j-2}\le 0,\ \text{for } j=r+2,r+3,\dots,n+1,$$

$$\beta_jt_1t_2+\beta_{j-1}(t_1-t_2)-\beta_{j-2}\ge 0,\ \text{for } j=2,3,\dots,r+1,$$

$$\beta_jt_1t_2+\beta_{j-1}(t_1-t_2)-\beta_{j-2}\le 0,\ \text{for } j=r+2,r+3,\dots,n+1,$$

for some λ with $1\le r\le n$, where we take $\alpha_{n+1}=\beta_{n+1}=0$, then all zeros of p lie in

$$|z|\le\frac{t_1}{|a_n|}\left\{\left(\frac{2\alpha_r}{t_1^{n-r}}-\alpha_n\right)+\frac{1}{t_1^n}(|\alpha_0|-\alpha_0)\right\}+\frac{t_2}{|a_n|}\left\{\frac{2\alpha_{r+1}}{t_1^{n-r-1}}+\frac{1}{t_1^n}(|\alpha_0|-\alpha_0)\right\}$$

$$+\frac{t_1}{|a_n|}\left\{\left(\frac{2\beta_r}{t_1^{n-r}}-\beta_n\right)+\frac{1}{t_1^n}(|\beta_0|-\beta_0)\right\}+\frac{t_2}{|a_n|}\left\{\frac{2\beta_{r+1}}{t_1^{n-r-1}}+\frac{1}{t_1^n}(|\beta_0|-\beta_0)\right\}.$$

Singh and Shah remarked that Theorem 5.12 reduces to Dewan and Bidkham's Theorem 4.1 when each coefficient is real and $t_2=0$, and further reduces to Theorem 1.3 when $r=n$ and $a_0\ge 0$. In the same paper, Singh and Shah [59] also presented a related generalization of Theorem 4.9 due to Aziz and Zargar.

In 2013, Singh and Shah [60] gave another result related to Theorem 5.10:

Theorem 5.13. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$. If $t_1 > t_2 \geq 0$, can be found such that for $j = 2, 3, \ldots, n$ we have*

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0,$$

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0,$$

and for some real λ_1 and λ_2 we have

$$(\alpha_n + \lambda_1)(t_1 - t_2) - \alpha_{n-1} \geq 0, \text{ and}$$

$$(\beta_n + \lambda_2)(t_1 - t_2) - \beta_{n-1} \geq 0,$$

then all zeros of p lie in

$$\left| z + \frac{(\lambda_1 + i\lambda_2)(t_1 - t_2)}{a_n} \right| \leq R,$$

where

$$R = \{[(\alpha_n+\lambda_1)+(\beta_n+\lambda_2)](t_1-t_2)+(\alpha_n+\beta_n)t_2-(\alpha_1+\beta_1)t_2/t_1^{n-1}-(\alpha_0+\beta_0)/t_1^{n-1}$$

$$+(|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|)/t_1^n + (|\alpha_0| + |\beta_0|)t_2/t_1^n\}/|a_n|.$$

With $t_2 = 0$ in Theorem 5.13, one easily gets as corollary the following:

Corollary 5.14. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$. If $t > 0$, can be found such that*

$$t^n(\alpha_n + \lambda_1) \geq t^{n-1}\alpha_{n-1} \geq t^{n-2}\alpha_{n-1} \geq \cdots \geq t\alpha_1 \geq \alpha_0$$

and

$$t^n(\beta_n + \lambda_2) \geq t^{n-1}\beta_{n-1} \geq t^{n-2}\beta_{n-1} \geq \cdots \geq t\beta_1 \geq \beta_0$$

for some real λ_1 and λ_2, then all zeros of p lie in

$$\left| z + \frac{(\lambda_1 + i\lambda_2)t}{a_n} \right| \leq R$$

where

$$R = t\{(\alpha_n + \lambda_1) + (\beta_n + \lambda_2) - [\alpha_0 + \beta_0 - |\alpha_0| - |\beta_0|]/t^n\}/|a_n|.$$

Among the many results listed above which overlap with Corollary 5.14, included Joyal, Labelle, and Rahman's Theorem 2.3, which follows from the corollary when $\lambda_1 = \lambda_2 = 0$ and $t = 1$.

Many of the results above, such as Corollary 5.14, involve the parameter $t > 0$ in such a way that coefficient a_j (or possibly its real part, imaginary part, or modulus) is multiplied by t^j and then involved in some type of monotonicity condition. Such a result often will follow from a simpler result which does not involve parameter t by applying the simpler result to polynomial $p(tz)$. Recently, Gulzar, Liman, and Shah [39] introduced a condition on the coefficients which is somewhat more subtle and does not yield a result which will easily follow from a simpler theorem. They required the parameter t to follow a pattern similar to that given in Corollary 5.14, but only for *some* of the coefficients. They prove:

Theorem 5.15. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients satisfying*

$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq t a_k \leq t^2 a_{k+1} \leq \cdots \leq t^{n-k} a_{n-1} \leq t^{n-k+1} a_n$$

for some $t > 0$ and $1 \leq k \leq n$. Then all the zeros of p lie in

$$|z + (t-1)| \leq \frac{a_n - a_0 + |a_0| + (t-1)\left\{\sum_{j=k}^{n}(a_j + |a_j|) - |a_n|\right\}}{|a_n|}.$$

With $k = n$ and $t = \lambda$, Theorem 5.15 implies Aziz and Zargar's Theorem 4.8. With $k = 1$, the hypotheses of Theorem 5.15 is similar to several of the results above (though there is no resulting reversal in the monotonicity hypothesis).

Choo and Choi [18] gave an interesting result in 2011 related to the hypothesis of monotonicity of the coefficients in the Eneström-Kakeya Theorem. They allowed one coefficient, say a_k, to violate the monotonicity condition and then constrained the deviation of a_k from a_{k-1} and a_{k+1} such that the zeros of the polynomial would still lie in $|z| \leq 1$:

Theorem 5.16. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $a_{r_i} = a_{r_i+1} = \cdots = a_{r_{i+1}-1}$ for $i = 0, 1, 2, \ldots, m$, where $r_0 = 0 < r_1 < r_2 < \cdots < r_m < r_{m+1} = n+1$. Suppose that for some $0 \leq k \leq m-1$,*

$$a_{r_m} > a_{r_{m-1}} > \cdots > a_{r_{k+1}} > a_{r_{k-1}} > a_{r_{k-2}} > \cdots > a_{r_n} > 0$$

and let

$$\rho = \max\left\{\frac{a_{r_{m-1}}}{a_{r_m}}, \frac{a_{r_{m-2}}}{a_{r_{m-1}}}, \ldots, \frac{a_{r_{k+1}}}{a_{r_{k+2}}}, \frac{a_{r_{k-1}}}{a_{r_{k+1}}}, \frac{a_{r_{k-2}}}{a_{r_{k-1}}}, \ldots, \frac{a_{r_0}}{a_{r_1}}\right\}.$$

If $\rho < 1$, then p has all its zeros in the disk $|z| \leq 1$ provided $a_{r_{k-1}} - \epsilon_1 \leq a_{r_k} \leq a_{r_{k+1}} + \epsilon_2$ where

$$\epsilon_1 = \frac{(1-\rho)R_1}{1 + \rho + (1-\rho)(r_{k+1} - r_k - 1)} \text{ and } \epsilon_2 = \frac{(1-\rho)R_2}{1 + \rho + (1-\rho)(r_{k+1} - r_k - 1)},$$

$$R_1 = a_{r_m} + a_{r_{m-1}} + \cdots + a_{r_{k+1}} + a_{r_{k-2}} + \cdots + a_{r_1} + a_0$$
$$-\{(n - r_m)a_{r_m} + (r_m - r_{m-1} - 1)a_{r_{m-1}} + \cdots + (r_{k+2} - r_{k+1} - 1)a_{r_{k+1}}$$
$$+(r_{k+1} - r_{k-1} - 2)a_{r_{k-1}} + (r_{k-1} - r_{k-2} - 1)a_{r_{k-2}} + \cdots + (r_2 - r_1 - 1)a_{r_1} + (r_1 - 1)a_0\},$$

and

$$R_2 = a_{r_m} + a_{r_{m-1}} + \cdots + a_{r_{k+2}} + a_{r_{k-1}} + \cdots + a_{r_1} + a_0$$
$$-\{(n - r_m)a_{r_m} + (r_m - r_{m-1} - 1)a_{r_{m-1}} + \cdots + (r_{k+2} - r_k - 2)a_{r_{k+1}}$$
$$+(r_k - r_{k-1} - 1)a_{r_{k-1}} + (r_{k-1} - r_{k-2} - 1)a_{r_{k-2}} + \cdots + (r_2 - r_1 - 1)a_{r_1} + (r_1 - 1)a_0\}.$$

Choo and Choi gave examples of polynomials illustrating their result. In particular, they gave $P(z) = 3.6z^6 + 5z^5 + 4z^4 + 3.2z^3 + 2.5z^2 + 2z + 1.5$ as an example of a polynomial which violates the monotonicity condition of Eneström-Kakeya, but which still has its zeros in $|z| \leq 1$. Coefficient a_6 violates the monotonicity condition; the authors computed $\epsilon_1 = 1.4667$ and observed that $a_6 \geq a_5 - \epsilon_1$, thus indicating that the hypotheses of their theorem are satisfied. In the same issue of the same journal, Choo and Choi [17] introduced the following generalization of the Eneström-Kakeya Theorem:

Theorem 5.17. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying*

$$a_0 \leq a_1 \leq \cdots \leq a_{n-k-1} \leq \lambda a_{n-k} \leq a_{n-k+1} \leq \cdots \leq a_n$$

for some real λ, then the zeros of p lie in $|z| \geq R$ where

$$R = \frac{|a_0|}{|a_n| + a_n + |(\lambda - 1)a_{n-k}| + (\lambda - 1)a_{n-k} - a_0} \text{ if } a_{n-k-1} \geq a_{n-k},$$

and

$$R = \frac{|a_0|}{|a_n| + a_n + |(\lambda - 1)a_{n-k}| + (1 - \lambda)a_{n-k} - a_0} \text{ if } a_{n-k} \geq a_{n-k+1}.$$

In the same paper, Choo and Choi gave a similar result by hypothesizing that the real parts of p satisfy the conditions of Theorem 5.17 and that the imaginary parts are monotone increasing. They also gave the following result which has a hypothesis concerning the moduli of the coefficients of p:

Theorem 5.18. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ where $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β for $j = 0, 1, 2, \ldots, n$. If*

$$|a_0| \leq |a_1| \leq \cdots \leq |a_{n-k-1}| \leq \lambda|a_{n-k}| \leq |a_{n-k+1}| \leq \cdots \leq |a_n|$$

for some $\lambda > 0$, then the zeros of p lie in $|z| \geq R$ where

$$R = \frac{|a_0|}{(|a_n| + (\lambda - 1)|a_{n-k}|)(\cos\alpha + \sin\alpha) - |a_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|}$$

if $|a_{n-k-1}| \geq |a_{n-k}|$, and

$$R = \frac{|a_0|}{(|a_n| + (1 - \lambda)|a_{n-k}|)(\cos\alpha + \sin\alpha) - |a_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|}$$

if $|a_{n-k}| \geq |a_{n-k+1}|$.

In 2012, Aziz and Zargar [10] modified the hypotheses of their own 1996 result, Theorem 4.8, and proved the following three theorems.

Theorem 5.19. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients such that for some $\lambda \geq 1$ and $0 < \rho \leq 1$ we have*

$$0 \leq \rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq \lambda a_n,$$

then all the zeros of p lie in $|z + \lambda - 1| \leq \lambda + 2a_0(1-\rho)/a_n$.

Theorem 5.20. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients such that for some $0 < \rho \leq 1$ and some $0 \leq k \leq n$ we have*

$$\rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n,$$

then all the zeros of p lie in

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|}\{2a_k - a_{n-1} + (2-\rho)|a_0| - \rho a_0\}.$$

Theorem 5.21. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients such that for some $0 < \rho \leq 1$ and some $0 \leq k \leq n$ we have*

$$\rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq \lambda a_n,$$

then all the zeros of p lie in

$$|z| \leq \frac{2a_k - a_n + (2-\rho)|a_0| + \rho a_0}{|a_n|}.$$

Aziz and Zargar [10] also showed that each of these implies Theorem 2.3 of Joyal, Labelle, and Rahman, and hence it is a generalization of the Eneström-Kakeya Theorem.

Recently, Gulzar [33] (see also [31]) proved:

Theorem 5.22. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\rho\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_{n-1} \leq \sigma + \alpha_n$$

for some $\sigma \geq 0$ and $0 < \rho \leq 1$, then the zeros of p lie in

$$\left| z + \frac{\sigma}{\alpha_n} \right| \leq \frac{\sigma + \alpha_n - \rho(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}.$$

Under the same hypotheses, Gulzar [38] gave an inner radius for a zero-free region for p as given in Theorem 5.22 by showing that p has no zeros in $|z| \leq |a_0|/\{2\sigma + a_n + |a_n| - \rho(a_0 + |a_0|) + |a_0|\}$. With similar monotonicity hypotheses concerning the coefficients, but with the added factor σ as in Theorem 5.22 and with $\rho = 1$, Liman, Shah, and Ahmad [49] gave additional related results in a 2013 publication.

In a result related to Choo and Choi's Theorem 5.17 and to Theorem 5.22, Gulzar [32] proved:

Theorem 5.23. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_0 \le \alpha_1 \le \ldots \le \alpha_{n-k-1} \le \lambda\alpha_{n-k} \le \alpha_{n-k+1} \le \cdots \le \alpha_{n-1} \le \sigma + \alpha_n$$

for some $\sigma \ge 0$ and real λ, where $\alpha_{n-k} \neq 0$, then the zeros of p lie in

$$\left| z + \frac{\sigma}{a_n} \right| \le \frac{\sigma + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}$$

if $\alpha_{n-k-1} > \alpha_{n-k}$, and the zeros lie in

$$\left| z + \frac{\sigma}{a_n} \right| \le \frac{\sigma + \alpha_n + (1 - \lambda)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}$$

if $\alpha_{n-k} > \alpha_{n-k+1}$.

With $\sigma = 0$ and each $\beta_j = 0$ in Gulzar's Theorem 5.23, one can produce an annulus (centered at origin) containing all the zeros of the polynomial where the inner radius is given by Choo and Choi's Theorem 5.17.

In [35], Gulzar combines the hypotheses of his own Theorem 5.22 and Theorem 5.23 (with parameters ρ, σ, and λ) to present three generalizations of the Eneström-Kakeya Theorem (with hypotheses on (1) the real part, (2) the imaginary part, and (3) the modulus of the coefficients).

6 Related Results

In this survey, we have have tried to present results that put a restriction on the modulus of the zeros of a polynomial *explicitly* in terms of the coefficients of the polynomial, as the original Eneström-Kakeya Theorem does. There are other results related to the Eneström-Kakeya Theorem which we have not yet been able to mention due to restrictions in the length of this chapter, but will now describe them briefly.

We say "explicitly" in the previous paragraph, because there are a number of results which restrict the modulus of the zeros of a polynomial, but the restrictions are given indirectly in the sense of being given by a root of a polynomial itself. This type of result was first given by Cauchy [13] , who proved the following:

Theorem 6.1. *Let $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$, be a complex polynomial. Then all the zeros of p(z) lie in the disk*

$$\{z : |z| < \eta\} \subset \{z : |z| < 1 + A\}, \tag{1}$$

where

$$A = \max_{0 \le j \le n-1} |a_j|,$$

and η is the unique positive root of the real-coefficient polynomial

$$Q(x) = x^n - |a_{n-1}|x^{n-1} - |a_{n-2}|x^{n-2} - \cdots - |a_1|x - |a_0|. \tag{2}$$

Govil and Rahman [30] also gave this type of result, and the same is stated as follows:

Theorem 6.2. *Let $p(z) = \sum_{j=0}^{n} a_j z^j \not\equiv 0$ be a polynomial of degree n with complex coefficients such that for some $a > 0$, we have $a^n|a_0| \leq a^{n-1}|a_1| \leq \ldots \leq a|a_{n-1} \leq |a_n|$. Then all the zeros of p lie in $|z| \leq \left(\frac{1}{a}\right) M$ where M is the greatest positive root of the trinomial equation $x^{n+1} - 2x^n + 1 = 0$.*

Related results concerning the location of the zeros of a polynomial have also been presented by Aziz and Mohammad [7], Sun and Hsieh [61], Affane-Aji, Agarwal, and Govil [1], Affane-Aji, Biaz and Govil [2], Choo [15], Choo and Choi [17], Dalal and Govil [19], Gulzar [34, 36], and Gilani [27].

The hypotheses of the following result, due to Jain [43] in 1988, are very much in the spirit of the Eneström-Kakeya Theorem, although the conclusion involves the size of the real part of the zeros instead of the modulus:

Theorem 6.3. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ where $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β, and $j = 0, 1, 2, \ldots, n$. If $0 < |a_0| \leq |a_1| \leq \cdots \leq |a_{n-1}| \leq |a_n| = 1$. Then all the zeros of p lie in the vertical strip $\{z : -\max\{1, \delta_2\} \leq Re(z) \leq \delta_2\}$ where $\delta_1 = [(1-\alpha_1)+\{(1-\alpha_2)^2+4M\}^{1/2}]/2$, $\delta_2 = [-(1-\alpha_1)+\{(1-\alpha_2)^2+4M\}^{1/2}]/2$, and $M = (|a_1|-|a_n|)(\cos\alpha+\sin\alpha)+2\sin\alpha\left(\sum_{j=2}^{n}|a_j|\right)+|a_n|$.*

In the same paper, Jain gave a result by putting the monotonicity hypothesis on the real parts of the coefficients. He also presented the corresponding result which put restrictions on the imaginary parts of the zeros of the polynomial p. Also, Jain [45] in 1993 gave a result which restricts the real part of the zeros , but with no condition on the coefficients (and hence not really related to the Eneström-Kakeya Theorem). It seems that this type of approach to the restriction of the zeros has been relatively little studied.

The techniques used in proving many of the theorems above can also be used to establish a bound on the moduli of the zeros of an analytic function which has a related monotonicity-type condition on the coefficients of its series representation. For example, Govil and Rahman [30] included the following result in their 1968 paper which was primarily devoted to polynomials:

Theorem 6.4. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an analytic function in $|z| \leq 1$. Suppose $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β for $j = 0, 1, 2, \ldots$ and $|a_0| \geq |a_1| \geq |a_2| \geq \cdots$. Then the zeros of f lie in $|z| \geq \left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|\right\}^{-1}$.*

Also closely related to the results of this survey is the following which is due to Aziz and Mohammad [6] and appeared in 1980.

Theorem 6.5. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an analytic function in $|z| \leq t$. Suppose $0 < a_0 \leq ta_1 \leq t^2 a_2 \leq \cdots$. Then all the zeros of f lie in $|z| \geq t$.*

Related results have been presented by Krishnaiah [48], Aziz and Shah [8], Lin, Huang, Cao, and Gardner [50], Shah and Liman [56], Choo [16], and Gulzar [37].

A natural question is: "Is the Eneström-Kakeya Theorem sharp?" In other words, is there a polynomial p satisfying the hypotheses of the the Eneström-Kakeya Theorem for which there is a zero of modulus 1 (thus indicating that the given bound cannot be improved)? For $p_n(z) = 1 + z + z^2 + \cdots + z^n$, the zeros of p_n are the $(n+1)^{\text{th}}$ roots of unity, $\cos\theta + i\sin\theta$ for $\theta = 2k\pi/(n+1)$ for $k = 1, 2, \ldots, n$. Therefore the Eneström-Kakeya Theorem (Theorem 1.3) is sharp. In fact, this example shows that the alternate version of the Eneström-Kakeya Theorem (Theorem 1.4) is also sharp.

However it is possible to sharpen the Eneström-Kakeya Theorem by taking away a part of the unit disk that does not contain the zeros of the polynomial, and this has been done, among others, by Govil and Rahman (see Theorem 5 in [30]), and by Rubinstein (see Corollary 1 in [55]).

In 1912-13, Hurwitz [42] characterized polynomials for which the Eneström-Kakeya Theorem is sharp. In 1979, Anderson, Saff, and Varga [4] gave a proof (and correction) of Hurwitz's result based on matrix methods. In a sense, their result states that a polynomial satisfying the hypotheses of the Eneström-Kakeya Theorem has a zero of modulus 1 only if the polynomial has p_n as a factor for some n (this is an oversimplification of their result, but somewhat reflects the importance of this result). An interesting corollary to their main theorem deals with the version of the Eneström-Kakeya theorem as stated in Theorem 1.4:

Corollary 6.6. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real and positive coefficients, then all the zeros of p lie in the annulus $R_1 \leq |z| \leq R_2$ where $R_1 = \min_{0 \leq j \leq n-1} a_j/a_{j+1}$ and $R_2 = \max_{0 \leq j \leq n-1} a_j/a_{j+1}$. If $R_1 < R_2$, then it is not possible for p to simultaneously have zeros on $|z| = R_1$ and on $|z| = R_2$.*

In a related result, Anderson, Saff, and Varga [5] in 1980 introduced a "generalized Eneström-Kakeya functional" and established a result concerning the location of zeros of polynomials and showed that their result is asymptotically sharp.

Appendix

REMARK ON A THEOREM ON THE ROOTS OF THE EQUATION
$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ WHERE
ALL COEFFICIENTS ARE REAL AND POSITIVE
by

G. Eneström, Stockholm, Sweden

Tôhoku Mathematical Journal, **18** (1920), 34–36

A translation of "Remarque sur un théorème relatif aux racines de l'equation $a_nx^n + a_{n-1}x^{n-1} + \cdots a_1x + a_0 = 0$ où tous les coefficientes a sont réels et positifs" by G. Eneström. Translated by Robert Gardner.

In 1912 M. S. Kakeya demonstrated in a paper in this journal that the absolute value of each root of the equation above [in the title] is between the smallest and largest values of

$$\frac{a_{n-1}}{a_n}, \frac{a_{n-2}}{a_{n-1}}, \cdots, \frac{a_0}{a_1},$$

and therefore for [positive] $a_n > a_{n-1} > \cdots > a_0$, the absolute value of each root is less than 1.

This theorem has already been proposed and demonstrated by me in 1893 in a footnote to a problem on pension funds. This problem leads us to the equation

$$k^{s-1} + a_1k^{s-2} + \cdots + a_{s-2}k + a_{s-1} = 0 \qquad \text{(A)}$$

where all of these coefficients are real and positive and for which

$$1 > a_1 \geq a_2 \geq \cdots \geq a_{s-1}.$$

The reference cited in the footnote is written in Swedish and at the request of Mr. Hayashi I now translate verbatim the part about the roots of this equation.

Define α_1 as the smallest of the quantities

$$a_1, \frac{a_2}{a_1}, \frac{a_3}{a_2}, \cdots, \frac{a_{s-1}}{a_{s-2}}$$

Härledning af en allmän formel för antalet pensionärer, som vid en godtyeklig tidpunkt förefinnas inom en sluten pensionslcassa; *Öfversigt af Vetenskaps-Akademiens Förhandlingar* (Stockholm), 50, 1893, p. 405–415. The resulting theorem was stated by me, also in *L'intermédiaire des Mathématiciens* 2, 1895, p. 418, and in *Jahrbuch über die Fortschritte der Mathematik* 25 (1893–1894), p. 360, and also mentions the problem of the theory of pensions to which I alluded in the text.

and it is then evident from this definition of α_1 that

$$a_{q+1} - \alpha_1 a_q \geq 0 \;\; (q = 0, 1, 2, \ldots, s-2; a_0 = 1).$$

Multiplication of equation (A) by $k - \alpha_1$, results in

$$k^s + (a_1 - \alpha_1)k^{s-1} + (a_2 - \alpha_1 a_1)k^{s-2} + \cdots + (a_{s-1} - \alpha_1 a_{s-2})k - \alpha_1 a_{s-1} = 0 \quad \text{(B)}$$

and if we substitute $\rho(\cos\phi + i\sin\phi)$ for k, where ρ is the absolute value of k, then ρ and ϕ must satisfy the equations

$$\begin{aligned}
&\rho^s \cos s\phi + (a_1 - \alpha_1)\rho^{s-1}\cos(s-1)\phi + (a_2 - \alpha_1 a_1)\rho^{s-2}\cos(s-2)\phi \\
&\qquad\qquad + \cdots + (a_{s-1} - \alpha_1 a_{s-2})\rho\cos\phi - \alpha_1 a_{s-1} = 0, \\
&\rho^s \sin s\phi + (a_1 - \alpha_1)\rho^{s-1}\sin(s-1)\phi + (a_2 - \alpha_1 a_1)\rho^{s-2}\sin(s-2)\phi \\
&\qquad\qquad + \cdots + (a_{s-1} - \alpha_1 a_{s-2})\rho\sin\phi = 0. \qquad \text{(C)}
\end{aligned}$$

We now show that if $\rho < \alpha_1$, equation (C) can not hold, regardless of the value of ϕ. Indeed, all coefficients $a_1 - \alpha_1, a_2 - \alpha_1 a_1, \ldots, a_{s-1} - \alpha_1 a_{s-2}$ are positive, so the left side can not be greater than

$$\rho^s + (a_1 - \alpha_1)\rho^{s-1} + (a_2 - \alpha_1 a_1)\rho^{s-2} + \cdots + (a_{s-1} - \alpha_1 a_{s-2})\rho - \alpha_1 a_{s-1}$$

and this expression can be written as

$$\rho^{s-1}(\rho - \alpha_1) + a_1\rho^{s-2}(\rho - \alpha_1) + \cdots + a_{s-2}\rho(\rho - \alpha_1) + a_{s-1}(\rho - \alpha_1),$$

which is negative if $\rho < \alpha_1$. The left side of equation (C) is therefore negative for $\rho < \alpha_1$. It follows that the absolute value of each root of equation (A) is greater than or equal to α_1.

In a similar way we can show that with α_2 as the largest of the quantities

$$a_1, \frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_{s-2}}{a_{s-1}},$$

the absolute value of each root of the equation (A) must be less than or equal to α_2. For this proof, replace k with k^{s-1} [in equation (A)]. Then multiply the new equation by $k - 1/\alpha_2$ and we easily find that the absolute value of k can never be less than $1/\alpha_2$, from which it follows immediately that the value of k can not be greater than α_2. We now have

$$\alpha_1 \leq |k_i| \leq \alpha_2, \;\; (i = 0, 1, 2, \ldots, s-1).$$

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