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THE NUMBER OF ZEROS OF A POLYNOMIAL IN A DISK

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Abstract. In this paper, we put restrictions on the coefficients of polynomials and give bounds concerning the number of zeros in a specific region. Our results generalize a number of previously known theorems, as well as implying a number of new corollaries with hypotheses concerning monotonicity of the real and imaginary parts of the coefficients.

1. Introduction

There is a large body of research on the location in the complex plane of some or all of the zeros of a polynomial in terms of the coefficients of the polynomial. For a survey, see Part II of Rahman and Schmeisser's *Analytic Theory of Polynomials* [9]. The famous Eneström-Kakeya Theorem states that for polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$, if the coefficients satisfy $0 \leqslant a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$, then all the zeros of p lie in $|z| \leqslant 1$ (see section 8.3 of [9]). In connection with the location of zeros of an analytic function $f(z) = \sum_{j=0}^{\infty} a_j z^j$, where $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$, Aziz and Mohammad imposed the condition $0 < \alpha_0 \leqslant t\alpha_1 \leqslant \cdots \leqslant t^k \alpha_k \geqslant t^{k+1}\alpha_{k+1} \geqslant \cdots$ (and a similar condition on the β_j s) [1]. These types of conditions have also been put on the coefficients of polynomials in order to get a restriction on the location of zeros [4]. In this paper, we impose these types of restrictions on the coefficients of polynomials in order to count the number of zeros in a certain region.

In Titchmarsh's classic *The Theory of Functions*, he states and proves the following (see page 171 of the second edition) [10].

THEOREM A. *Let $F(z)$ be analytic in $|z| \leqslant R$. Let $|F(z)| \leqslant M$ in the disk $|z| \leqslant R$ and suppose $F(0) \neq 0$. Then for $0 < \delta < 1$ the number of zeros of $F(z)$ in the disk $|z| \leqslant \delta R$ is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|F(0)|}.$$

By putting a restriction on the coefficients of a polynomial similar to that of the Eneström-Kakeya Theorem, Mohammad used a special case of Theorem A to prove the following [7].

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THEOREM B. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *be such that* $0<a_0\leqslant a_1\leqslant a_2\leqslant\cdots\leqslant a_{n-1}\leqslant a_n$. *Then the number of zeros in* $|z|\leqslant\frac{1}{2}$ *does not exceed*

$$1+\frac{1}{\log 2}\log\left(\frac{a_n}{a_0}\right).$$

In her dissertation work, Dewan weakens the hypotheses of Theorem B and proves the following two results for polynomials with complex coefficients [2, 6].

THEOREM C. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *be such that* $|arg(a_j)-\beta|\leqslant\alpha\leqslant\pi/2$ *for all* $1\leqslant j\leqslant n$ *and some real* α *and* β, *and* $0<|a_0|\leqslant|a_1|\leqslant|a_2|\leqslant\cdots\leqslant|a_{n-1}|\leqslant|a_n|$. *Then the number of zeros of* p *in* $|z|\leqslant 1/2$ *does not exceed*

$$\frac{1}{\log 2}\log\frac{|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|}{|a_0|}.$$

THEOREM D. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j)=\alpha_j$ *and* $Im(a_j)=\beta_j$ *for all* j *and* $0<\alpha_0\leqslant\alpha_1\leqslant\alpha_2\leqslant\cdots\leqslant\alpha_{n-1}\leqslant\alpha_n$, *then the number of zeros of* p *in* $|z|\leqslant 1/2$ *does not exceed*

$$1+\frac{1}{\log 2}\log\frac{\alpha_n+\sum_{j=0}^{n}|\beta_j|}{|\alpha_0|}.$$

Pukhta generalized Theorems C and D by finding the number of zeros in $|z|\leqslant\delta$ for $0<\delta<1$ [8]. The next theorem deals with a monotonicity condition on the moduli of the coefficients.

THEOREM E. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *be such that* $|arg(a_j)-\beta|\leqslant\alpha\leqslant\pi/2$ *for all* $1\leqslant j\leqslant n$ *and some real* α *and* β, *and* $0<|a_0|\leqslant|a_1|\leqslant|a_2|\leqslant\cdots\leqslant|a_{n-1}|\leqslant|a_n|$. *Then the number of zeros of* p *in* $|z|\leqslant\delta$, $0<\delta<1$, *does not exceed*

$$\frac{1}{\log 1/\delta}\log\frac{|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|}{|a_0|}.$$

Pukhta also gave a result which involved a monotonicity condition on the real part of the coefficients [8]. Though the proof presented by Pukhta is correct, there was a slight typographical error in the statement of the result as it appeared in print. The correct statement of the theorem is as follows.

THEOREM F. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *be such that* $|arg(a_j)-\beta|\leqslant\alpha\leqslant\pi/2$ *for all* $1\leqslant j\leqslant n$ *and some real* α *and* β, *and* $0<\alpha_0\leqslant\alpha_1\leqslant\alpha_2\leqslant\cdots\leqslant\alpha_{n-1}\leqslant\alpha_n$. *Then* *the number of zeros of* p *in* $|z| \leqslant \delta$, $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{2\left(\alpha_n + \sum_{j=0}^{n} |\beta_j|\right)}{|a_0|}.$$

In this paper, we further weaken the hypotheses of the above results and prove the following.

THEOREM 1. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where for some* $t > 0$ *and some* $0 \leqslant k \leqslant n$,

$$0 < |a_0| \leqslant t|a_1| \leqslant t^2|a_2| \leqslant \cdots \leqslant t^{k-1}|a_{k-1}| \leqslant t^k|a_k| \\ \geqslant t^{k+1}|a_{k+1}| \geqslant \cdots \geqslant t^{n-1}|a_{n-1}| \geqslant t^n|a_n|$$

and $|\arg a_j - \beta| \leqslant \alpha \leqslant \pi/2$ *for* $1 \leqslant j \leqslant n$ *and for some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leqslant \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $M = |a_0|t(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha + |a_n|t^{n+1}(1 + \sin\alpha - \cos\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|t^{j+1}$.

Notice that when $t = 1$ in Theorem 1, we get the following.

COROLLARY 1. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where for some* $t > 0$ *and some* $0 \leqslant k \leqslant n$,

$$0 < |a_0| \leqslant |a_1| \leqslant |a_2| \leqslant \cdots \leqslant |a_{k-1}| \leqslant |a_k| \geqslant |a_{k+1}| \geqslant \cdots \geqslant |a_{n-1}| \geqslant |a_n|$$

and $|\arg a_j - \beta| \leqslant \alpha \leqslant \pi/2$ *for* $1 \leqslant j \leqslant n$ *and for some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leqslant \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

where $M = |a_0|(1-\cos\alpha-\sin\alpha)+2|a_k|\cos\alpha+|a_n|(1+\sin\alpha-\cos\alpha)+2\sin\alpha \sum_{j=0}^{n-1} |a_j|$.

With $k = n$ in Corollary 1, the hypothesis becomes $0 < |a_0| \leqslant |a_1| \leqslant \cdots \leqslant |a_n|$, and the value of M becomes $|a_0|(1-\cos\alpha-\sin\alpha)+|a_n|(1+\sin\alpha+\cos\alpha)+2\sin\alpha \sum_{j=0}^{n-1} |a_j|$. Since $0 \leqslant \alpha \leqslant \pi/2$, we have $1 - \cos\alpha - \sin\alpha \leqslant 0$. So the value of M given by Theorem 1 is less than or equal to $|a_n|(1 + \sin\alpha + \cos\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|$, and Theorem 1 implies Theorem E.

THEOREM 2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leqslant j \leqslant n$. *Suppose that for some* $t > 0$ *and some* $0 \leqslant k \leqslant n$ *we have*

$$0 \neq \alpha_0 \leqslant t\alpha_1 \leqslant t^2\alpha_2 \leqslant \cdots \leqslant t^{k-1}\alpha_{k-1} \leqslant t^k\alpha_k \geqslant t^{k+1}\alpha_{k+1} \geqslant \cdots \geqslant t^{n-1}\alpha_{n-1} \geqslant t^n\alpha_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leqslant \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|},$$

where $M = (|\alpha_0| - \alpha_0)t + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=0}^{n} |\beta_j| t^{j+1}$.

Notice that with $t = 1$ in Theorem 2, we get the following.

COROLLARY 2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leqslant j \leqslant n$. *Suppose we have*

$$0 \neq \alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{k-1} \leqslant \alpha_k \geqslant \alpha_{k+1} \geqslant \cdots \geqslant \alpha_{n-1} \geqslant \alpha_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leqslant \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| - \alpha_0) + 2\alpha_k + (|\alpha_n| - \alpha_n) + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

With $k = n$ and $0 < \alpha_0$ in Corollary 2, the hypothesis becomes $0 < \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n$ and the value of M becomes $2(\alpha_n + \sum_{j=0}^{n} \beta_j)$; therefore Theorem F follows from Corollary 2. With $\beta_j = 0$ for $1 \leqslant j \leqslant n$ and $\delta = 1/2$, Corollary 2 reduces to a result of Dewan and Bidkham [3].

As an example, consider the polynomial $p(z) = (z+0.1)^2(z+10)^2 = 1 + 20.2z + 104.01z^2 + 20.2z^3 + z^4$. With $\alpha_0 = \alpha_4 = 1$, $\alpha_1 = \alpha_3 = 20.2$, $\alpha_2 = 104.01$, and each $\beta_j = 0$, we see that Corollary 2 applies to p with $k = 2$, however none of Theorems B through F apply to p. With $\delta = 0.1$, Corollary 2 implies that the number of zeros in $|z| \leqslant \delta = 0.1$ is less than $\frac{1}{\log(1/0.1)} \log \frac{2(104.01)}{1} \approx 2.318$, which implies that p has at most two zeros in $|z| \leqslant 0.1$, and of course p has exactly two zeros in this region. We also observe that Theorem A applies to p, but requires that we find a bound for $|p(z)|$ for $|z| = R = 1$; this fact makes it harder to determine the bound given by the conclusion of Theorem A, as opposed to the other results mentioned above which give bounds in terms of the coefficients of p. Since all the coefficients of p in this example are positive, it is quite easy to find this maximum, and Theorem A also implies that p has at most two zeros in $|z| \leqslant \delta = 0.1$.

THEOREM 3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leqslant j \leqslant n$. *Suppose that for some* $t > 0$, *for some* $0 \leqslant k \leqslant n$ *we have*

$$0 \neq \alpha_0 \leqslant t\alpha_1 \leqslant t^2\alpha_2 \leqslant \cdots \leqslant t^{k-1}\alpha_{k-1} \leqslant t^k\alpha_k \geqslant t^{k+1}\alpha_{k+1} \geqslant \cdots \geqslant t^{n-1}\alpha_{n-1} \geqslant t^n\alpha_n,$$

and for some $0 \leqslant \ell \leqslant n$ *we have*

$$\beta_0 \leqslant t\beta_1 \leqslant t^2\beta_2 \leqslant \cdots \leqslant t^{\ell-1}\beta_{\ell-1} \leqslant t^{\ell}\beta_{\ell} \geqslant t^{\ell+1}\beta_{\ell+1} \geqslant \cdots \geqslant t^{n-1}\beta_{n-1} \geqslant t^n\beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leqslant \delta t$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|},$$

where $M = (|\alpha_0| - \alpha_0)t + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + (|\beta_0| - \beta_0)t + 2\beta_\ell t^{\ell+1} + (|\beta_n| - \beta_n)t^{n+1}$.

Theorem 3 gives several corollaries with hypotheses concerning monotonicity of the real and imaginary parts. For example, with $t = 1$ and $k = \ell = n$ we have the hypotheses that $0 \neq \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n$ and $\beta_0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_n$, resulting in the following.

COROLLARY 3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leqslant j \leqslant n$. *Suppose that we have*

$$0 \neq \alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{n-1} \leqslant \alpha_n \text{ and } \beta_0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_{n-1} \leqslant \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leqslant \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| - \alpha_0) + (|\alpha_n| + \alpha_n) + (|\beta_0| - \beta_0) + (|\beta_n| + \beta_n)}{|a_0|}.$$

With $t = 1$ and $k = \ell = 0$, Theorem 3 gives the following.

COROLLARY 4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leqslant j \leqslant n$. *Suppose that we have*

$$0 \neq \alpha_0 \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{n-1} \geqslant \alpha_n \text{ and } \beta_0 \geqslant \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_{n-1} \geqslant \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leqslant \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| + \alpha_0) + (|\alpha_n| - \alpha_n) + (|\beta_0| + \beta_0) + (|\beta_n| - \beta_n)}{|a_0|}.$$

With $t = 1$, we can let $k = n$ and $\ell = 0$ (or $k = 0$ and $\ell = n$), Theorem 3 gives the next two results.

COROLLARY 5. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leqslant j \leqslant n$. *Suppose that we have*

$$0 \neq \alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{n-1} \leqslant \alpha_n \text{ and } \beta_0 \geqslant \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_{n-1} \geqslant \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leqslant \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| - \alpha_0) + (|\alpha_n| + \alpha_n) + (|\beta_0| + \beta_0) + (|\beta_n| - \beta_n)}{|a_0|}.$$

COROLLARY 6. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $0 \leqslant j \leqslant n$. *Suppose that we have*

$$0 \neq \alpha_0 \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{n-1} \geqslant \alpha_n \text{ and } \beta_0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_{n-1} \leqslant \beta_n.$$

Then for $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leqslant \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{(|\alpha_0| + \alpha_0) + (|\alpha_n| - \alpha_n) + (|\beta_0| - \beta_0) + (|\beta_n| + \beta_n)}{|a_0|}.$$

2. Proofs of the Theorems

The following is due to Govil and Rahman and appears in [5].

LEMMA 1. *Let* $z, z' \in \mathbb{C}$ *with* $|z| \geqslant |z'|$. *Suppose* $|\arg z^* - \beta| \leqslant \alpha \leqslant \pi/2$ *for* $z^* \in \{z, z'\}$ *and for some real* α *and* β. *Then*

$$|z - z'| \leqslant (|z| - |z'|) \cos\alpha + (|z| + |z'|) \sin\alpha.$$

We now give proofs of our results.

Proof of Theorem 1. Consider

$$\begin{aligned} F(z) &= (t-z)P(z) = (t-z)\sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n} (a_j t z^j - a_j z^{j+1}) \\ &= a_0 t + \sum_{j=1}^{n} a_j t z^j - \sum_{j=1}^{n} a_{j-1} z^j - a_n z^{n+1} \\ &= a_0 t + \sum_{j=1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1}. \end{aligned}$$

For $|z| = t$ we have

$$\begin{aligned} |F(z)| &\leqslant |a_0| t + \sum_{j=1}^{n} |a_j t - a_{j-1}| t^j + |a_n| t^{n+1} \\ &= |a_0| t + \sum_{j=1}^{k} |a_j t - a_{j-1}| t^j + \sum_{j=k+1}^{n} |a_{j-1} - a_j t| t^j + |a_n| t^{n+1} \\ &\leqslant |a_0| t + \sum_{j=1}^{k} \left\{ (|a_j| t - |a_{j-1}|) \cos\alpha + (|a_{j-1}| + |a_j| t) \sin\alpha \right\} t^j \\ &\quad + \sum_{j=k+1}^{n} \left\{ (|a_{j-1}| - |a_j| t) \cos\alpha + (|a_j| t + |a_{j-1}|) \sin\alpha \right\} t^j + |a_n| t^{n+1} \end{aligned}$$

by Lemma 1 with $z = a_j t$ and $z' = a_{j-1}$ when $1 \leqslant j \leqslant k$, and with $z = a_{j-1}$ and $z' = a_j t$ when $k+1 \leqslant j \leqslant n$

$$
\begin{aligned}
&= |a_0|t + \sum_{j=1}^{k} |a_j| t^{j+1} \cos\alpha - \sum_{j=1}^{k} |a_{j-1}| t^j \cos\alpha + \sum_{j=1}^{k} |a_{j-1}| t^j \sin\alpha \\
&\quad + \sum_{j=1}^{k} |a_j| t^{j+1} \sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}| t^j \cos\alpha - \sum_{j=k+1}^{n} |a_j| t^{j+1} \cos\alpha \\
&\quad + \sum_{j=k+1}^{n} |a_j| t^{j+1} \sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}| t^j \sin\alpha + |a_n| t^{n+1} \\
&= |a_0|t + |a_k| t^{k+1} \cos\alpha + \sum_{j=1}^{k-1} |a_j| t^{j+1} \cos\alpha - |a_0| t \cos\alpha - \sum_{j=1}^{k-1} |a_j| t^{j+1} \cos\alpha \\
&\quad + |a_0| t \sin\alpha + \sum_{j=1}^{k-1} |a_j| t^{j+1} \sin\alpha + |a_k| t^{k+1} \sin\alpha + \sum_{j=1}^{k-1} |a_j| t^{j+1} \sin\alpha \\
&\quad + |a_k| t^{k+1} \cos\alpha + \sum_{j=k+1}^{n-1} |a_j| t^{j+1} \cos\alpha - |a_n| t^{n+1} \cos\alpha - \sum_{j=k+1}^{n-1} |a_j| t^{j+1} \cos\alpha \\
&\quad + |a_n| t^{n+1} \sin\alpha + \sum_{j=k+1}^{n-1} |a_j| t^{j+1} \sin\alpha + |a_k| t^{k+1} \sin\alpha \\
&\quad + \sum_{j=k+1}^{n-1} |a_j| t^{j+1} \sin\alpha + |a_n| t^{n+1} \\
&= |a_0|t + |a_k| t^{k+1} \cos\alpha - |a_0| t \cos\alpha + |a_0| t \sin\alpha + |a_k| t^{k+1} \sin\alpha \\
&\quad + 2\sum_{j=1}^{k-1} |a_j| t^{j+1} \sin\alpha + |a_k| t^{k+1} \cos\alpha - |a_n| t^{n+1} \cos\alpha \\
&\quad + |a_n| t^{n+1} \sin\alpha + |a_k| t^{k+1} \sin\alpha + 2\sum_{j=k+1}^{n-1} |a_j| t^{j+1} \sin\alpha + |a_n| t^{n+1} \\
&= |a_0| t (1 - \cos\alpha - \sin\alpha) + |a_k| (2t^{k+1} \cos\alpha + 2t^{k+1} \sin\alpha) \\
&\quad + |a_n| t^{n+1} (1 + \sin\alpha - \cos\alpha) + 2\sum_{j=0}^{k-1} |a_j| t^{j+1} \sin\alpha + 2\sum_{j=k+1}^{n-1} |a_j| t^{j+1} \sin\alpha \\
&= |a_0| t (1 - \cos\alpha - \sin\alpha) + 2|a_k| t^{k+1} \cos\alpha + |a_n| t^{n+1} (1 + \sin\alpha - \cos\alpha) \\
&\quad + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| t^{j+1} \\
&= M.
\end{aligned}
$$

Now $F(z)$ is analytic in $|z| \leqslant t$, and $|F(z)| \leqslant M$ for $|z| = t$. So by Theorem A and the Maximum Modulus Theorem, the number of zeros of F (and hence of P) in $|z| \leqslant \delta t$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

The theorem follows. □

Proof of Theorem 2. As in the proof of Theorem 1,

$$F(z)=(t-z)P(z)=a_0t+\sum_{j=1}^{n}(a_jt-a_{j-1})z^j-a_nz^{n+1},$$

and so

$$\begin{aligned}F(z)&=(\alpha_0+i\beta_0)t+\sum_{j=1}^{n}((\alpha_j+i\beta_j)t-(\alpha_{j-1}+i\beta_{j-1}))z^j-(\alpha_n+i\beta_n)z^{n+1}\\&=(\alpha_0+i\beta_0)t+\sum_{j=1}^{n}(\alpha_jt-\alpha_{j-1})z^j+i\sum_{j=1}^{n}(\beta_jt-\beta_{j-1})z^j-(\alpha_n+i\beta_n)z^{n+1}\end{aligned}$$

For $|z|=t$ we have

$$\begin{aligned}|F(z)|&\leqslant(|\alpha_0|+|\beta_0|)t+\sum_{j=1}^{n}|\alpha_jt-\alpha_{j-1}|t^j+\sum_{j=1}^{n}(|\beta_j|t+|\beta_{j-1}|)t^j+(|\alpha_n|+|\beta_n|)t^{n+1}\\&=(|\alpha_0|+|\beta_0|)t+\sum_{j=1}^{k}(\alpha_jt-\alpha_{j-1})t^j+\sum_{j=k+1}^{n}(\alpha_{j-1}-\alpha_jt)t^j+\sum_{j=1}^{n-1}|\beta_j|t^{j+1}\\&\quad+|\beta_n|t^{n+1}+|\beta_0|t+\sum_{j=1}^{n-1}|\beta_j|t^{j+1}+(|\alpha_n|+|\beta_n|)t^{n+1}\\&=|\alpha_0|t+\sum_{j=1}^{k-1}\alpha_jt^{j+1}+\alpha_kt^{k+1}-\alpha_0t-\sum_{j=1}^{k-1}\alpha_jt^{j+1}+\alpha_kt^{k+1}\\&\quad+\sum_{j=k+1}^{n-1}\alpha_jt^{j+1}-\alpha_nt^{n+1}-\sum_{j=k+1}^{n-1}\alpha_jt^{j+1}+2\sum_{j=0}^{n}|\beta_j|t^{j+1}+|\alpha_n|t^{n+1}\\&=(|\alpha_0|-\alpha_0)t+2\alpha_kt^{k+1}+(|\alpha_n|-\alpha_n)t^{n+1}+2\sum_{j=0}^{n}|\beta_j|t^{j+1}\\&=M.\end{aligned}$$

The result now follows as in the proof of Theorem 1. □

Proof of Theorem 3. As in the proof of Theorem 2,

$$F(z)=(\alpha_0+i\beta_0)t+\sum_{j=1}^{n}(\alpha_jt-\alpha_{j-1})z^j+i\sum_{j=1}^{n}(\beta_jt-\beta_{j-1})z^j-(\alpha_n+i\beta_n)z^{n+1}.$$

For $|z|=t$ we have

$$|F(z)|\leqslant(|\alpha_0|+|\beta_0|)t+\sum_{j=1}^{n}|\alpha_jt-\alpha_{j-1}|t^j+\sum_{j=1}^{n}|\beta_jt-\beta_{j-1}|t^j+(|\alpha_n|+|\beta_n|)t^{n+1}$$

$$
\begin{aligned}
&= (|\alpha_0|+|\beta_0|)t+\sum_{j=1}^{k}|\alpha_j t-\alpha_{j-1}|t^j+\sum_{j=k+1}^{n}|\alpha_j t-\alpha_{j-1}|t^j\\
&\quad+\sum_{j=1}^{\ell}|\beta_j t-\beta_{j-1}|t^j+\sum_{j=\ell+1}^{n}|\beta_j t-\beta_{j-1}|t^j+(|\alpha_n|+|\beta_n|)t^{n+1}\\
&= (|\alpha_0|+|\beta_0|)t+\sum_{j=1}^{k}(\alpha_j t-\alpha_{j-1})t^j+\sum_{j=k+1}^{n}(\alpha_{j-1}-\alpha_j t)t^j\\
&\quad+\sum_{j=1}^{\ell}(\beta_j t-\beta_{j-1})t^j+\sum_{j=\ell+1}^{n}(\beta_{j-1}-\beta_j t)t^j+(|\alpha_n|+|\beta_n|)t^{n+1}\\
&= |\alpha_0|t+\sum_{j=1}^{k-1}\alpha_j t^{j+1}+\alpha_k t^{k+1}-\alpha_0 t-\sum_{j=1}^{k-1}\alpha_j t^{j+1}+\alpha_k t^{k+1}+\sum_{j=k+1}^{n-1}\alpha_j t^{j+1}\\
&\quad-\alpha_n t^{n+1}-\sum_{j=k+1}^{n-1}\alpha_j t^{j+1}+|\alpha_n|t^{n+1}+|\beta_0|t+\sum_{j=1}^{\ell-1}\beta_j t^{j+1}+\beta_\ell t^{\ell+1}\\
&\quad-\beta_0 t-\sum_{j=1}^{\ell-1}\beta_j t^{j-1}+\beta_\ell t^{\ell+1}+\sum_{j=\ell+1}^{n-1}\beta_j t^{j+1}-\beta_n t^{n+1}-\sum_{j=\ell+1}^{n-1}\beta_j t^{j+1}+|\beta_n|t^{n+1}\\
&= (|\alpha_0|-\alpha_0)t+2\alpha_k t^{k+1}+(|\alpha_n|-\alpha_n)t^{n+1}+(|\beta_0|-\beta_0)t+2\beta_\ell t^{\ell+1}\\
&\quad+(|\beta_n|-\beta_n)t^{n+1}\\
&= M.
\end{aligned}
$$

The result now follows as in the proof of Theorem 1. □

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