Decompositions, Packings, and Coverings of the Complete Digraph with Orientations of $K_3 \cup \{e\}$

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Abstract. There are eight orientations of the complete graph on three vertices with a pendant edge, $K_3 \cup \{e\}$. Two of these are 3-circuits with a pendant arc and the other six are transitive triples with a pendant arc. Necessary and sufficient conditions are given for decompositions, packings, and coverings of the complete digraph with each of these eight orientations of $K_3 \cup \{e\}$.

1 Introduction

A $G$-decomposition of a graph $H$ is a set $\{g_1, g_2, \ldots, g_n\}$ of subgraphs of $H$ (called blocks) such that $g_i \cong G$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^n E(g_i) = E(H)$. A $G$-decomposition of $H$ where $G$ and $H$ are digraphs is similarly defined (with arc sets replacing edge sets). Several decompositions of the complete graph $K_v$ and the complete digraph $D_v$ have been explored. In particular, a Steiner triple system of order $v$ is equivalent to a $K_3$-decomposition of $K_v$ and such systems exist if and only if $v \equiv 1$ or 3 (mod 6) [12]. A Mendelsohn triple system is equivalent to a 3-circuit ($C_3$) decomposition of $D_v$ and exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [9]. A directed triple system is equivalent to a transitive triple ($T$, see Figure 1) decomposition of $D_v$ and exists if and only if $v \equiv 0$ or 1 (mod 3) [8]. Also of relevance to our results are decompositions of $K_v$ into copies of $K_3$ with a pendant edge (the graph $L$ of Figure 1). Bermond and Schönhem showed that such decompositions exist if and only if $v \equiv 0$ or 1 (mod 8) [2].

A maximum $G$-packing of graph $H$ is a set $\{g_1, g_2, \ldots, g_n\}$ of subgraphs of $H$ (called blocks) such that $g_i \cong G$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^n g_i \subset H$, and $|E(H) \setminus \bigcup_{i=1}^n E(g_i)|$ is minimum. The leave of the packing is the set $E(H) \setminus \bigcup_{i=1}^n E(g_i)$. A maximum $G$-packing of $H$
where $G$ and $H$ are digraphs is similarly defined (with arc sets replacing edge sets). Maximum $K_3$-packings of $K_v$ were explored by Schönheim [10]. Maximum 3-circuit and transitive triple packings of $D_v$ were addressed in [5].

A minimum $G$-covering of graph $H$ is a set $\{g_1, g_2, \ldots, g_n\}$ of subgraphs of $H$ (called blocks) such that $g_i \cong G$ for $i \in \{1, 2, \ldots, n\}$, $H \subset \bigcup_{i=1}^{n} g_i$, and $|\bigcup_{i=1}^{n} E(g_i) \setminus E(H)|$ is minimum (the graph $\bigcup_{i=1}^{n} g_i$ may not be simple and $\bigcup_{i=1}^{n} E(g_i)$ may be a multiset). A minimum $G$-covering of $H$ where $G$ and $H$ are digraphs is similarly defined (with arc sets replacing edge sets). The padding of the covering is the multiset $\bigcup_{i=1}^{n} E(g_i) \setminus E(H)$. Minimum $K_3$-coverings of $K_v$ were explored by Fort and Hedlund [3]. Minimum 3-circuit and transitive triple coverings of $D_v$ were addressed in [5].

We note that $K_3$-decompositions of $K_v$ were followed by decompositions of $D_v$ with orientations of $K_3$. Thus, a natural follow-up to the work of [2] would be to consider orientations of graphs of order four or less. Because of this, we are motivated to consider decompositions, packings, and coverings of $D_v$ with copies of digraph $G$ where $G$ is an orientation of $L = K_3 \cup \{e\}$ (see Figure 2). We denote the orientations of $L = K_3 \cup \{e\}$ given in Figure 2 as $[a, b, c; d]_{m_1}$, $[a, b, c; d]_{m_2}$, $[a, b, c; d]_{d_1}$, ..., $[a, b, c; d]_{d_6}$, respectively. The purpose of this paper is to give necessary and sufficient conditions for decompositions, packings, and coverings of $D_v$ with each of the eight orientations of $L = K_3 \cup \{e\}$.

2 Decompositions

We note that since each of these orientations has four arcs, it is necessary that $|A(D_v)| \equiv 0 \pmod{4}$ for the existence of a decomposition of $D_v$ into one of the digraphs of Figure 2. Hence $v \equiv 0$ or 1 (mod 4) is necessary in all cases.

The wheel, denoted $W_n$, is the graph containing a cycle on $n$ vertices such that every vertex in the cycle is adjacent to a center vertex, $\infty$. We will denote the wheel $W_n$ with center $\infty$ and cycle $(0, a, 2a, \ldots, (n-1)a)$ by $W_n(\infty : a)$. Note that $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$. This can
be extended to a digraph by replacing each edge with a forward arc and a backward arc.

\[ \begin{array}{cccc}
  b & c & b & c \\
  a & d & a & d \\
  m_1 & m_2 & d_1 & d_2 \\
\end{array} \]

\[ \begin{array}{cccc}
  b & c & b & c \\
  a & d & a & d \\
  d_3 & d_4 & d_5 & d_6 \\
\end{array} \]

Figure 2. The eight orientations of \( L = K_3 \cup \{e\} \).

The circulant, denoted \( C_n(S) \), has vertex set \( V(C_n(S)) = \{0, 1, \ldots, n-1\} \). Two vertices \( u \) and \( v \) are adjacent if and only if \( |u - v|_n \in S \), where \( |x|_n = \min\{x \mod n, n - x \mod n\} \). The directed circulant will have a forward arc and a backward arc for each of these edges.

A graceful labeling on a graph \( G \) with \( q \) edges is an injective mapping \( f \) from \( V(G) \) to \( \{0, 1, \ldots, q\} \) such that the edge labels defined by \( f'(u, v) = |f(u) - f(v)| \) satisfy \( f'(E) = \{1, 2, \ldots, q\} \) [6, 11]. We note that wheels have graceful labelings [4, 7]. This being the case, there exists a \( W_p \)-decomposition of \( C_n(1, 2, \ldots, 2p) \) where \( n \geq 4p + 1 \) [1].

**Theorem 2.1** An \( m_1 \)-decomposition of \( D_v \) and an \( m_2 \)-decomposition of \( D_v \) each exist if and only if \( v \equiv 0 \) or \( 1 \) (mod 4).

**Proof.** We note that \( v \equiv 0 \) or \( 1 \) (mod 4) is necessary by the above comments. Further note that there exists an \( m_1 \)-decomposition of the directed wheel \( W_p \), where \( p \geq 3 \). This decomposition is given by the set of blocks \( \{[j, \infty, j+1; j-1]_{m_1} \mid j = 0, 1, \ldots, p-1\} \) where the numerical vertex labels are reduced modulo \( p \).

**Case 1.** Suppose \( v \equiv 0 \) (mod 4), say \( v = 4k + 4 \) where \( k \geq 3 \). We note that \( D_{4k+4} = W_{4k+3}(\infty; 2k+1) \cup C_{4k+3}(1, 2, \ldots, 2k) \) where \( V(D_{4k+4}) = \)
{0, 1, 2, ..., 4k + 2, \infty}. There exists an \(m_1\)-decomposition of \(W_{4k+3}\) and \(C_{4k+3}(1, 2, \ldots, 2k)\) for \(k \geq 3\) by the above comments.

For \(v = 4\), \(D_4 \cong W_3\) and a decomposition of \(W_3\) is given above.

For \(v = 8\), the decomposition is given by the set of blocks \(\{[j, \infty, j + 2; j + 1]_{m_1}, [j, j + 1, j + 3; j + 4]_{m_1} \mid j = 0, 1, \ldots, 6\}\) where vertex labels are reduced modulo 7.

For \(v = 12\), the decomposition is given by the set of blocks \(\{[j+5, \infty, j+10; j]_{m_1}, [j, j+1, j+3; j+7]_{m_1}, [j, j+3, j+1; j+4]_{m_1} \mid j = 0, 1, \ldots, 10\}\) where numerical vertex labels are reduced modulo 11.

Case 2. Suppose \(v \equiv 1 \pmod{4}\), say \(v = 4k + 1\), where \(k \geq 3\). Since \(W_k\) is graceful, there exists a decomposition of \(K_{4k+1}\) by the above comments. It follows that the directed wheel \(W_k\) decomposes the directed complete graph \(D_{4k+1}\).

For \(v = 5\), the decomposition is given by the set of blocks \(\{[4, 0, 1; 3]_{m_1}, [4, 3, 0; 2]_{m_1}, [3, 2, 0; 1]_{m_1}, [1, 0, 2; 4]_{m_1}, [2, 3, 1; 4]_{m_1}\}\).

For \(v = 9\), the decomposition is given by the set of blocks \(\{[j, j+1, j+3; j+5]_{m_1}, [j, j+3, j+1; j+4]_{m_1} \mid j = 0, 1, \ldots, 8\}\) where vertex labels are reduced modulo 9.

Since \(m_2\) is the converse of \(m_1\), the construction of an \(m_2\)-decomposition of \(D_v\) will similarly follow.

Theorem 2.2 A \(d_1\)-decomposition of \(D_v\) and a \(d_2\)-decomposition of \(D_v\) each exist if and only if \(v \equiv 0\) or \(1 \pmod{4}\).

Proof. The necessary condition follows as in Theorem 2.1. We now construct a \(d_1\)-decomposition of \(D_v\) for each \(v \equiv 0\) or \(1 \pmod{4}\) and, since \(d_2\) is the converse of \(d_1\), the construction of a \(d_2\)-decomposition of \(D_v\) will similarly follow.

Case 1. Suppose \(v \equiv 1 \pmod{12}\), say \(v = 12k + 1\). Consider the set of blocks: \(\{[j, 6k - i + j, 12k - 2i + j; 3k + 1 + i + j]_{d1}, [j, 5k - i + j, 10k - 2i + j; 8k + 1 + 2i + j]_{d1} \mid i = 0, 1, \ldots, k-1, j = 0, 1, \ldots, 12k\}\)
\(\cup\{[j, k - 1 - i + j, 12k - 3 - 2i + j; 2k + 2 + i + j]_{d1} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 12k\}\) \(\cup\{[j, k + j, 12k - 1 + j; k + 1 + j]_{d1} \mid j = 0, 1, \ldots, 12k\}\).

Case 2. Suppose \(v \equiv 5 \pmod{12}\), say \(v = 12k + 5\). Consider the set of blocks: \(\{[j, 6k + 2 - i + j, 12k + 4 - 2i + j; 3k + 1 + i + j]_{d1}, [j, 5k + 1 - i + j, 10k + 2 - 2i + j; 8k + 5 + 2i + j]_{d1} \mid i = 0, 1, \ldots, k-1, j = 0, 1, \ldots, 12k + 4\}\)
\(\cup\{[j, k - 1 - i + j, 12k + 1 - 2i + j; 2k + 2 + i + j]_{d1} \mid i = 0, 1, \ldots, k-2, j = 0, 1, \ldots, 12k + 4\}\)
\(\cup\{[j, 5k + 2 + j, 10k + 4 + j; 4k + 1 + j]_{d1} \mid j = 0, 1, \ldots, 12k + 4\} \cup\{[j, 5k + 2 + j, 10k + 4 + j; 4k + 1 + j]_{d1} \mid j = 0, 1, \ldots, 12k + 4, \text{ omit if } k = 0\}.)

Case 3. Suppose \(v \equiv 9 \pmod{12}\), say \(v = 12k + 9\). Consider the set of
blocks: \{[j, 6k + 4 - i + j, 12k + 8 - 2i + j; 3k + 4 + i + j]_A, [j, 5k + 3 - i + j, 10k + 6 - 2i + j; 8k + 7 + 2i + j]_A, [j, k - i + j, 12k + 5 - 2i + j; 2k + 4 + i + j]_A \mid i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 12k + 8\} \cup \{[j, 5k + 4 + j, 10k + 8 + j; 8k + 6 + j]_A, [j, k + 1 + j, 12k + 7 + j; k + 2 + j]_A \mid j = 0, 1, \ldots, 12k + 8\}.

In each of Cases 1–3, the given set of blocks forms a decomposition of \(D_v\) where \(V(D_v) = \{0, 1, \ldots, v - 1\}\) and vertex labels in the blocks are reduced modulo \(v\).

**Case 4.** Suppose \(v \equiv 0 \pmod{4}\), say \(v = 4k\). Consider the set of blocks: \{\([j, 2 + j, \infty; 1 + j]_A \mid j = 0, 1, \ldots, 4k - 2\}\} \cup \{\{[j, k + 1 - i + j, k + 2 + i + j; 2k + 1 + 2i + j]_A \mid i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 4k - 2\}\}. In Case 4, the given set of blocks forms a decomposition of \(D_v\) where \(V(D_v) = \{\infty, 0, 1, \ldots, v - 2\}\) and numerical vertex labels in the blocks are reduced modulo \(v - 1\). 

**Corollary 2.3** A \(d_3\)-decomposition of \(D_v\) and a \(d_4\)-decomposition of \(D_v\) each exist if and only if \(v \equiv 0 \text{ or } 1 \pmod{4}\).

**Proof.** The necessary condition follows as in Theorem 2.1. In the case \(v \equiv 1 \pmod{4}\), blocks for such a \(d_3\)-decomposition can be constructed from the \(d_1\)-decomposition of Theorem 2.2 by replacing every block of the form \([j, a + j, b + j; c + j]_A\) with a block of the form \([a + j, b + j, j; a + c + j]_A\). In the case \(v \equiv 0 \pmod{4}\), blocks for such a \(d_3\)-decomposition can be constructed from the \(d_1\)-decomposition of Theorem 2.2 by replacing every block of the form \([j, a + j, b + j; c + j]_A\) with a block of the form \([a + j, b + j, j; a + c + j]_A\) and by replacing every block of the form \([j, a + j, \infty; c + j]_A\) with a block of the form \([a + j, \infty; a + c + j]_A\).

Since \(d_4\) is the converse of \(d_3\), the construction of a \(d_4\)-decomposition of \(D_v\) will similarly follow.

**Corollary 2.4** A \(d_5\)-decomposition of \(D_v\) and a \(d_6\)-decomposition of \(D_v\) each exist if and only if \(v \equiv 1 \pmod{4}\).

**Proof.** As in Theorem 2.1, one necessary condition is that \(v \equiv 0 \text{ or } 1 \pmod{4}\). Notice that the vertices of \(d_5\) are of in-degrees 0, 0, 2, and 2. Therefore another necessary condition for a \(d_5\)-decomposition on \(D_v\) (and similarly for a \(d_6\)-decomposition of \(D_v\)) is that each vertex of \(D_v\) is of even in-degree — that is, \(v\) must be odd. Therefore \(v \equiv 1 \pmod{4}\) is necessary.

Blocks for such a \(d_5\)-decomposition of \(D_v\) can be constructed from the \(d_1\) system of Theorem 2.2 by replacing every block of the form \([j, a + j, b + j; c + j]_A\) with a block of the form \([b + j, a + j, j; b + c + j]_A\).

Since \(d_6\) is the converse of \(d_5\), the construction of a \(d_6\)-decomposition of \(D_v\) will similarly follow.
3 Packings

We now give necessary and sufficient conditions for the packing of $D_v$ with each of the eight orientations of $L$.

**Theorem 3.1** A maximum $m_1$-packing of $D_v$ with leave $L$ satisfies

(i) $|A(L)| = 0$ if $v \equiv 0$ or $1 \pmod{4}$,

(ii) $|A(L)| = 6$ if $v = 3$, and $|A(L)| = 2$ if $v \equiv 2$ or $3 \pmod{4}$.

Maximum $m_2$-packings of $D_v$ satisfy the same conditions.

**Proof.** If $v \equiv 0$ or $1 \pmod{4}$, then there is a decomposition by Theorem 2.1 and the result follows. If $v \equiv 2$ or $3 \pmod{4}$, then $|A(D_v)| = 2 \pmod{4}$, and so a packing with leave $L$ where $|A(L)| = 2$ would be maximum.

**Case 1.** Let $v \equiv 3 \pmod{4}$, say $v = 4k + 3$ where $k \geq 4$. We note that:

$$D_{4k+3} = W_{4k+1}(\infty_1 : 2k - 1) \cup W_{4k+1}(\infty_2 : 2k) \cup C_{4k+1}(1, 2, \ldots, 2k - 2) \cup \{(\infty_1, \infty_2), (\infty_2, \infty_1)\}.$$

As shown in the proof of Theorem 2.1, there exists an $m_1$-decomposition of $W_{4k+1}$ and $W_{k-1}$ for $k \geq 4$. Since $W_{k-1}$ is graceful, there exists a $W_{k-1}$-decomposition of $C_{4k+1}(1, 2, \ldots, 2k - 2)$.

The result is trivial for $v = 3$.

For $v = 7$, we note that: $D_7 = W_5(\infty_1 : 1) \cup W_5(\infty_2 : 2) \cup \{ (\infty_1, \infty_2), (\infty_2, \infty_1) \}$.

For $v = 11$, the required packing is given by the set of blocks $\{ [j, j+1, \infty_1; j+7]_{m_1}, [j, j+5, \infty_2; j+3]_{m_1}, [j, j+3, j+1; j+5]_{m_1} | j = 0, 1, \ldots, 8 \}$ where numerical vertex labels are reduced modulo 9.

For $v = 15$, we note that: $D_{15} = W_9(\infty_1 : 5) \cup W_9(\infty_2 : 6) \cup C_{13}(1, 2, 3, 4) \cup \{ (\infty_1, \infty_2), (\infty_2, \infty_1) \}$. As above, there exists an $m_1$-decomposition of $W_{13}$. An $m_1$-decomposition of $C_{13}(1, 2, 3, 4)$ is given by the set of blocks $\{ [j, j+1, j+3; j+4]_{m_1}, [j, j+3, j+1; j+9]_{m_1} | j = 0, 1, \ldots, 12 \}$ where vertex labels are reduced modulo 13.

In each case above, the leave of the packing is $\{ (\infty_1, \infty_2), (\infty_2, \infty_1) \}$.

**Case 2.** Let $v \equiv 2 \pmod{4}$, say $v = 4k + 2$ where $k \geq 8$. We note that:

$$D_{4k+2} = D_7 \cup C_{4k-5}(1, 2, \ldots, 2k - 10) \cup \bigcup_{i=1}^{7} W_{4k-5}(\infty_1 : 2k - i - 2).$$

As above, there exists an $m_1$-decomposition of $W_{4k-5}$ and $C_{4k-5}(1, 2, \ldots, 2k - 10)$ for $k \geq 8$. Further, there exists a maximum $m_1$-packing of $D_7$ with leave size two, as shown above.
For $v = 6$, the required packing is given by the set of blocks $\{[0, 1, 5; 2]_{m_1}, [0, 5, 1; 3]_{m_1}, [4, 0, 2; 1]_{m_1}, [4, 1, 3; 0]_{m_1}, [5, 3, 2; 4]_{m_1}, [2, 3, 1; 5]_{m_1}, [3, 5, 4; 0]_{m_1}\}$. This packing has leave $\{4, 2\}, (2, 1)$.

For $v = 10$, the required packing is given by the set of blocks $\{[1 + j, \infty_1, j; 2 + j]_{m_1} \mid j = 0, 1, \ldots, 5\} \cup \{[2 + 3j, \infty_2, 5 + 3j; 6 + 3j]_{m_1} \mid j = 0, 1, 2, 3, 4\} \cup \{[2j, \infty_3, 2j + 2; 2j + 5]_{m_1} \mid i = 0, 1, \ldots, 6\} \cup \{[6, 0, 3; \infty_1]_{m_1}, [6, 3, \infty_2; 2]_{m_1}, [\infty_1, \infty_2, \infty_3; 0]_{m_1}, [\infty_2, \infty_1, \infty_3; 6]_{m_1}\}$, where all numerical vertex labels are reduced modulo 7. This packing has leave $\{(1, 0), (\infty_2, 2)\}$.

For $v = 14$, the required packing is given by the set of blocks $\{[3j + 3, \infty_1, 3j + 6]_{m_1} \mid j = 0, 1, \ldots, 9\} \cup \{[1 + 4j, \infty_2, 5 + 4j; 8 + 4j]_{m_1} \mid j = 0, 1, \ldots, 8\} \cup \{[2j, \infty_3, 2j + 2; 2j + 9]_{m_1}, [j, j + 1, j + 5; j + 10]_{m_1} \mid j = 0, 1, \ldots, 10\} \cup \{[8, 0, 4, \infty_1]_{m_1}, [8, 4, \infty_2; 1]_{m_1}, [\infty_1, \infty_2, \infty_3; 0]_{m_1}, [\infty_2, \infty_1, \infty_3; 8]_{m_1}\}$, where all numerical vertex labels are reduced modulo 11. The leave on this packing is $\{(3, 0), (\infty_2, 1)\}$.

For $v = 18$, the required packing is given by the set of blocks $\{[j + 1, \infty_1, j; 2 + 2j]_{m_1} \mid j = 0, 1, \ldots, 13\} \cup \{[6 + 7j, \infty_2, 13 + 7j; 14 + 7j]_{m_1} \mid j = 0, 1, \ldots, 12\} \cup \{[2j, \infty_3, 2j + 2; 2j + 13]_{m_1}, [j, j + 4, j + 9; j + 3]_{m_1}, [j, j + 9, j + 4; j + 12]_{m_1} \mid j = 0, 1, \ldots, 14\} \cup \{[14, 0, 7; \infty_1]_{m_1}, [14, 7, \infty_2; 6]_{m_1}, [\infty_1, \infty_2, \infty_3; 0]_{m_1}, [\infty_2, \infty_1, \infty_3; 14]_{m_1}\}$, where all numerical vertex labels are reduced modulo 15. The leave is $\{(1, 0), (\infty_2, 6)\}$.

For $v = 22$, we have $D_{22} = D_7 \cup_{i=1}^{5} W_{15}(\infty_1 : i)$. This has a maximum packing with leave size two by the above comments.

For $v = 26$, the required packing is given by the set of blocks $\{[j, j + 18, \infty_1; j + 4]_{m_1}, [j, j + 14, \infty_2; j + 6]_{m_1}, [j, j + 12, \infty_3; j + 8]_{m_1}, [j, j + 10, \infty_4; j + 11]_{m_1}, [j, j + 9, \infty_5; j + 12]_{m_1}, [j, j + 6, \infty_6; j + 14]_{m_1}, [j, j + 3, \infty_7; j + 15]_{m_1}, [j, j + 1, j + 3; j + 2]_{m_1} \mid j = 0, 1, \ldots, 18\}$, where all numerical vertex labels are reduced modulo 19. The remaining arcs are isomorphic to $D_7$, which has a maximum packing with leave size two by the above comments.

For $v = 30$, we have $D_{30} = D_7 \cup C_{23}(1, 2, 3, 4) \cup_{i=1}^{5} W_{23}(\infty_i : 4 + i)$. The required $m_1$-decomposition of $C_{23}(1, 2, 3, 4)$ is given by the set of blocks $\{[i, j + 1, j + 3; j + 4]_{m_1}, [j, j + 3, j + 1; j + 19]_{m_1} \mid j = 0, 1, \ldots, 22\}$, where all numerical labels on the vertices are reduced modulo 23. $W_{23}$ has an $m_1$-decomposition by the above comments. $D_7$ has a maximum $m_1$-packing with leave size two.

Since $m_2$ is the converse of $m_1$, the construction of an $m_2$-packing of $D_v$ will similarly follow.

\begin{theorem}
A maximum $d_1$-packing of $D_v$ with leave $L$ satisfies
\begin{itemize}
  \item [(i)] $|A(L)| = 0$ if $v \equiv 0 \text{ or } 1 \pmod{4}$,
  \item [(ii)] $|A(L)| = 6$ if $v \in \{3, 6\}$, and $|A(L)| = 2$ if $v \equiv 2 \text{ or } 3 \pmod{4}$, $v \notin \{3, 6\}$.
\end{itemize}
\end{theorem}
Maximum $d_2$-packings of $D_v$ satisfy the same conditions.

Proof. The necessary conditions follow as in Theorem 3.1. If $v \equiv 0$ or $1 \pmod{4}$, then there is a decomposition by Theorem 2.2 and the result follows.

Case 1. Suppose $v \equiv 2 \pmod{4}$, say $v = 8k + 2$ where $k \geq 1$. Consider the sets $A = \{[j, 5k - i + j, 5k + 2 + i + j; 2k + 3 - i + j]_{d_1} | j = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 8k - 2\}$ and $B = \{[j, 1 + j, \infty_1; 4 + j]_{d_1}, [j, 2 + j, \infty_2; 3 + j]_{d_1}, [j, 5 + j, \infty_3; 5k + 1 + j]_{d_1} | j = 0, 1, \ldots, 8k - 2\}$. Then $A \cup B \cup \{[\infty_2, \infty_1, \infty_3; 2]_{d_1}, [\infty_1, \infty_2, \infty_3; 3]_{d_1}, [0, 3, 2; \infty_2]_{d_1} \} \setminus \{[2, 3, \infty_1; 6]_{d_1}, [0, 2, \infty_2; 3]_{d_1}\},$ where $V(D_v) = \{\infty_1, \infty_2, \infty_3, 0, 1, \ldots, v - 4\}$ and numerical vertex labels are reduced modulo $8k - 1$, is a maximum $d_1$-packing of $D_v$ with leave $L$ where $A(L) = \{\infty_1, 2\}.$

The result is trivial when $v = 2$.

Case 2. Suppose $v \equiv 3 \pmod{4}$, say $v = 4k + 3$ where $k \geq 1$. Consider the sets $A = \{[j, k + 3 - i + j, 4k - 2i + j; 2k + 3 + 2i + j]_{d_1} | j = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 4k\}$ and $B = \{[j, 1 + 2i + j, \infty_1; 2 + 2i + j]_{d_1} | i = 0, 1, j = 0, 1, \ldots, 4k\}$ where $V(D_v) = \{\infty_1, \infty_2, 0, 1, \ldots, v - 3\}$ and numerical vertex labels are reduced modulo $4k + 1$. Then $A \cup B$ is a maximum $d_1$-packing of $D_v$ with leave $L$ where $A(L) = \{\infty_1, \infty_2, \infty_1\}.$

The result is trivial when $v = 3$.

Case 3. Suppose $v \equiv 6 \pmod{8}$, say $v = 8k + 6$ where $k \geq 1$. Consider the sets $A = \{[j, 5k + 3 - i + j, 5k + 5 + i + j; 2k + 4 - i + j]_{d_1} | j = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 8k + 2\} \cup [j, 3k + 4 - i + j, 3k + 5 + i + j; 6 + i + j]_{d_1} | i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 8k + 2\}$ and $B = \{[j, 1 + j, \infty_1; 4 + j]_{d_1}, [j, 2 + j, \infty_2; 3 + j]_{d_1}, [j, 5 + j, \infty_3; 5k + 4 + j]_{d_1} | j = 0, 1, \ldots, 8k + 2\}$. Then $A \cup B \cup \{[\infty_2, \infty_1, \infty_3; 2]_{d_1}, [\infty_1, \infty_2, \infty_3; 3]_{d_1}, [0, 3, 2; \infty_2]_{d_1} \} \setminus \{[2, 3, \infty_1; 6]_{d_1}, [0, 2, \infty_2; 3]_{d_1}\},$ where $V(D_v) = \{\infty_1, \infty_2, \infty_3, 0, 1, \ldots, v - 4\}$ and numerical vertex labels are reduced modulo $8k + 3$, is a maximum $d_1$-packing of $D_v$ with leave $L$ where $A(L) = \{\infty_1, 2\}.$

When $v = 6$, $|A(D_v)| = 30$ and a $d_1$-packing of $D_6$ could contain as many as seven copies of $d_1$. However, each vertex of $D_6$ is of in-degree 5 and $d_1$ contains a vertex of in-degree 3. Therefore the number of $d_1$s in a $d_1$-packing of $D_6$ cannot exceed the number of vertices in $D_6$—namely, six. So in a maximum $d_1$-packing of $D_6$ with leave $L$, we have $|A(L)| \geq 6$. A maximum packing is given by $\{[0, 2, 4; 3]_{d_1}, [1, 2, 3; 0]_{d_1}, [2, 4, 3; 1]_{d_1}, [3, 5, 1; 0]_{d_1}, [4, 5, 6; 3]_{d_1}, [5, 1, 0; 4]_{d_1}\}$ where $A(L) = \{(3, 5), (0, 2), (2, 5), (5, 2), (1, 4), (4, 1)\}$ and $|A(L)| = 6$.

Since $d_2$ is the converse of $d_1$, the construction of a $d_2$-packing of $D_v$ will similarly follow.

Corollary 3.3 A maximum $d_3$-packing of $D_v$ with leave $L$ satisfies
(i) $|A(L)| = 0$ if $v \equiv 0 \text{ or } 1 \pmod{4}$,

(ii) $|A(L)| = 6$ if $v = 3$, and $|A(L)| = 2$ if $v \equiv 2 \text{ or } 3 \pmod{4}$, $v \neq 3$.

Maximum $d_4$-packings of $D_v$ satisfy the same conditions.

**Proof.** The necessary conditions follow as in Theorem 3.1.

For $v \neq 6$, the blocks for such a $d_3$-packing of $D_v$ can be constructed from the $d_1$-packing $D_v$ of Theorem 3.2 by replacing every block of the form $[j, a + j, b + j; c + j]_{d_1}$ with a block of the form $[a + j, b + j, j; a + c + j]_{d_3}$, replacing every block of the form $[a, b, c; d]_{d_3}$ with a block of the form $[-a, \infty_1, -b; c - 2a]_{d_3}$, and then (1) when $v \equiv 2 \pmod{8}$ by replacing the two blocks $[2, \infty_2, 0; 5]_{d_3}$ and $[5, \infty_3, 0; 5k + 6]_{d_3}$ with the three blocks $[\infty_2, \infty_1, \infty_3; 2]_{d_3}$, $[\infty_3, \infty_1, \infty_2; 5]_{d_3}$, and $[5, 2, 0; 5k + 6]_{d_3}$, and (2) when $v \equiv 6 \pmod{8}$ by replacing the two blocks $[2, \infty_2, 0; 5]_{d_3}$ and $[5, \infty_3, 0; 5k + 9]_{d_3}$ with the three blocks $[\infty_2, \infty_1, \infty_3; 2]_{d_3}$, $[\infty_3, \infty_1, \infty_2; 5]_{d_3}$, and $[5, 2, 0; 5k + 9]_{d_3}$. In the case $v = 2 \pmod{4}$, this is a $d_3$-packing of $D_v$, where $V(D_v) = \{\infty_1, \infty_2, \infty_3, 0, 1, \ldots, v-4\}$, with leave $L$ where $A(L) = \{(\infty_2, 0), (\infty_3, 0)\}$. In the case $v \equiv 3 \pmod{4}$, this is a $d_3$-packing of $D_v$, where $V(D_v) = \{\infty_1, \infty_2, 0, 1, \ldots, v - 3\}$, with leave $L$ where $A(L) = \{(\infty_1, \infty_2), (\infty_2, \infty_1)\}$.

For $v = 6$, consider the set of blocks $\{[4, 1, 3; 0]_{d_3}, [4, 5, 2; 3]_{d_3}, [5, 3, 0; 1]_{d_3}, [3, 1, 2; 0]_{d_3}, [0, 5, 1; 2]_{d_3}, [1, 4, 0; 2]_{d_3}, [2, 5, 4; 0]_{d_3}\}$. This is a maximum $d_3$-packing of $D_6$ with leave $L$ where $A(L) = \{(2, 3), (3, 5)\}$.

Since $d_4$ is the converse of $d_3$, the construction of a $d_4$-packing of $D_v$ will similarly follow.

**Theorem 3.4** A maximum $d_5$-packing of $D_v$ with leave $L$ satisfies

(i) $|A(L)| = v$ if $v \equiv 0 \pmod{2}$,

(ii) $|A(L)| = 0$ if $v \equiv 1 \pmod{4}$, and

(iii) $|A(L)| = 6$ if $v = 3$, and $|A(L)| = 2$ if $v \equiv 3 \pmod{4}$, $v \geq 7$.

Maximum $d_6$-packings of $D_v$ satisfy the same conditions.

**Proof.** When $v \equiv 1 \pmod{4}$, a decomposition exists by Corollary 2.4 and $|A(L)| = 0$ in this case. Notice that the vertices of $d_5$ are of in-degrees 0, 0, 2, and 2. So when $v$ is even, a $d_5$-packing of $D_v$ will have a leave $L$ where the in-degree of each vertex of $L$ is odd. So for $v$ even, a $d_5$-packing of $D_v$ with leave $L$ where $|A(L)| = v$ would be maximum (and similarly for a $d_6$-packing of $D_v$). When $v \equiv 3 \pmod{4}$, $|A(D_v)| = 2 \pmod{4}$ and in this case a $d_5$-packing (and similarly for a $d_6$-packing) of $D_v$ with leave...
$L$ where $|A(L)| = 2$ would be maximum. In the following cases, we have $V(D_v) = \{0, 1, \ldots, v-1\}$.

**Case 1.** Suppose $v \equiv 0 \pmod{4}$. Consider $A \cup B$ where $A = \{[2j, 4k - 1 + 2j, 1 + 2j; 4k - 2 + 2j]_{4d} | j = 0, 1, \ldots, 2k - 1\}$ and $B = \{[j, 3k - 3 + j, 4k - 2 + j; 3k - 2 + j]_{4d}\} \cup \{[j, 2k - 1 + i + j, 2k + 2 + 2i + j; 2k - 3 - 2i + j]_{4d} | i = 0, 1, \ldots, k - 3, j = 0, 1, \ldots 4k - 1\}$ where vertex labels are reduced modulo $4k$. Then $A \cup B$ is a maximum $d_5$-packing of $D_v$ with leave $L$ where $A(L) = \{(j, j-1) | j = 0, 1, \ldots, 4k - 1\}$.

**Case 2.** Suppose $v \equiv 2 \pmod{4}$, say $v = 4k + 2$. Consider $\{[j, k + 2 + i + j, 1 + 2i + j; 2k + 2 + 2i]_{4d} | i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 4k + 1\}$ where vertex labels are reduced modulo $4k + 2$. This is a maximum $d_5$-packing of $D_v$ with leave $L$ where $A(L) = \{(j, j-1) | j = 0, 1, \ldots, 4k + 1\}$.

**Case 3.** Suppose $v \equiv 3 \pmod{4}$. Consider $A \cup B$ where $A = \{[2i, 4k + 2 + 2i, 1 + 2i; 4k + 1 + 2i]_{4d} | i = 0, 1, \ldots, 2k\} B = \{[j, 3k - 1 + j, 4k - 2 + j; 4k + j]_{4d} | j = 0, 1, \ldots, 4k - 2\} \cup \{[j, 2k + i + j, 2k + 4 + 2i + j; 2 + 2i + j]_{4d} | i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 4k - 2\}$ where vertex labels are reduced modulo $4k + 3$. Then $A \cup B$ is a maximum $d_5$-packing of $D_v$ with leave $L$ where $A(L) = \{(4k, 4k + 2), (4k + 1, 4k + 2\}$.

Since $d_6$ is the converse of $d_5$, the construction of a $d_6$-packing of $D_v$ will similarly follow.

## 4 Covering

We now give necessary and sufficient conditions for the covering of $D_v$ with each of the eight orientations of $L$.

**Theorem 4.1** A minimum $m_1$-covering of $D_v$, $v \geq 4$, with padding $P$ satisfies

1. $|A(P)| = 0$ if $v \equiv 0 \text{ or } 1 \pmod{4}$, and
2. $|A(P)| = 2$ if $v \equiv 2 \text{ or } 3 \pmod{4}$.

Minimum $m_2$-coverings of $D_v$ satisfy the same conditions.

**Proof.** If $v \equiv 0 \text{ or } 1 \pmod{4}$, then there is a decomposition by Theorem 2.1 and the result follows. If $v \equiv 2 \text{ or } 3 \pmod{4}$, then $|A(D_v)| \equiv 2 \pmod{4}$, and so a covering with padding $P$ where $|A(P)| = 2$ would be minimum.

**Case 1.** Let $v \equiv 2 \pmod{4}$, say $v = 4k + 2$ where $k \geq 5$. We note that:

$$D_{4k+2} = D_3 \cup C_{4k-1}(1, 2, \ldots, 2k-4) \cup \bigcup_{i=1}^3 W_{4k-1}((\infty_i : 2k - i)) .$$

As above, there exists an $m_1$-decomposition of $W_{4k-1}$ and $C_{4k-1}(1, 2, \ldots, 2k-4)$ for $k \geq 5$. The remaining arcs are covered by the set $\{[\infty_1, \infty_2, \infty_3; 0]_{m_1}, [\infty_3, \infty_2, \infty_1; 1]_{m_1}\}$. This covering has padding $\{(0, \infty_1), (1, \infty_3)\}$. 


For \( v = 6 \), the required covering is obtained from the packing in Theorem 3.1 along with the set \([2, 1, 4; 3]_{m_1}\). This covering has padding \((1, 4), (3, 2)\).

For \( v = 10 \), the required covering is obtained from the packing in Theorem 3.1 along with the set \([2, 1, 0; \infty]_{m_1}\). This covering has padding \((2, 1), (0, 2)\).

For \( v = 14 \), the required covering is obtained from the packing in Theorem 3.1 along with the set \([1, 3, 0; \infty]_{m_1}\). This covering has padding \((1, 3), (0, 1)\).

For \( v = 18 \), the required covering is obtained from the packing in Theorem 3.1 along with the set \([6, 1, 0; \infty]_{m_1}\). This covering has padding \((6, 1), (0, 6)\).

**Case 2.** Let \( v \equiv 3 \pmod{4} \), say \( v = 4k + 3 \) where \( k \geq 7 \). We note that:

\[
D_{4k+3} = \bigcup_{i=1}^{6} W_{4k-3}((\infty_i : 2k - 1 - i) \cup C_{4k-3}(1, 2, \ldots, 2k - 8)) \cup D_6.
\]

As shown in the proof of Theorem 2.1, there exists an \( m_1 \)-decomposition of \( W_{4k-3} \) and \( W_{k-5} \) for \( k \geq 7 \). Since \( W_{k-4} \) is graceful, there exists a \( W_{k-4} \)-decomposition of \( C_{4k-3}(1, 2, \ldots, 2k - 8) \). Further, there exists a minimum covering of \( D_6 \) as given above. Thus there exists a minimum covering of \( D_{4k+3} \) for \( k \geq 8 \) with padding \((\infty_1, \infty_4), (\infty_3, \infty_2)\).

For \( v = 7 \), the covering is given by the set of blocks \([0, 6, 1; 4]_{m_1}, [0, 1, 6; 2]_{m_1}, [5, 1, 3; 0]_{m_1}, [5, 0, 2; 1]_{m_1}, [4, 3, 0; 6]_{m_1}, [3, 4, 6; 0]_{m_1}, [3, 6, 2; 5]_{m_1}, [5, 2, 4; 6]_{m_1}, [4, 2, 1; 5]_{m_1}, [6, 3, 2; 5]_{m_1}, [1, 5, 2; 4]_{m_1}\). The padding is \((6, 3), (5, 3)\).

For \( v = 11 \), the covering is given by the set of blocks \([1 + 3i, \infty_1, 7 + 3i; 3 + 3i]_{m_1}, [2 + 3i, \infty_1, 8 + 3i; 4 + 3i]_{m_1} | i = 0, 1, 2] \cup \{|4 + 4i, \infty_2, 4i%; 5 + 4i|_{m_1} | i = 0, 1, \ldots, 7\} \cup \{|i, i+1, i+3; i+4|_{m_1} | i = 0, 1, \ldots, 8\} \cup \{|\infty_1, 6, 0; \infty_2]_{m_1}, [\infty_2, 5, 0; \infty_1]_{m_1}, [3, \infty_1, 0; 5]_{m_1}, [6, \infty_1, 3; 8]_{m_1}, [0, 3, 1; 2]_{m_1}\}, \) where all numerical vertex labels are reduced modulo 9. The padding is \((0, 3), (3, 1)\).

For \( v = 15 \), the covering is given by the set of blocks \([5i, \infty_1, 5i+5; 5i+8]_{m_1}, [6i, \infty_2, 6i+6; 6i+7]_{m_1} | i = 0, 1, \ldots, 11\} \cup \{|i, i+1, i+3; i+4|_{m_1}, [i, i+3, i+1; i+9]_{m_1} | i = 0, 1, \ldots, 12\} \cup \{|\infty_1, 0, 8; \infty_2]_{m_1}, [\infty_2, 0, 7; \infty_1]_{m_1}, [7, 3, 8; 1]_{m_1}\}, \) where all numerical vertex labels are reduced modulo 13. The padding is \((7, 3), (8, 7)\).

For \( v = 19 \), we note that: \( D_{19} = \bigcup_{i=1}^{6} W_{13}(\infty_i : i) \cup D_6. \) There exists an \( m_1 \)-decomposition of \( W_{13} \) by the above comments. Further, there exists a minimum \( m_1 \)-covering of \( D_6 \) by above.

For \( v = 23 \), the covering is given by the set of blocks \([10i, \infty_1, 10i+10; 10i+11]_{m_1}, [9i, \infty_2, 9i+9; 9i+12]_{m_1} | i = 0, 1, \ldots, 19\} \cup \{|\infty_1, 0, 11; \infty_2]_{m_1}, [\infty_2, 0, 12; \infty_1]_{m_1}, [11, 3, 12; 1]_{m_1}\}, \) where all numerical vertex labels are reduced modulo 21. The remaining arcs are isomorphic to \( C_{21}(1, 2, \ldots, 8) \), which has an \( m_1 \)-decomposition by the above comments. The padding is \((11, 3), (12, 11)\).
For \( v = 27 \), the covering is given by the set of blocks \( \{[12i, \infty_1, 12i + 12; 12i + 13]_{m_1}, [11i, \infty_2, 11i + 11; 11i + 14]_{m_1} \mid i = 0, 1, \ldots, 23\} \cup \{[\infty_1, 0, 13; \infty_2]_{m_1}, [\infty_2, 0, 14; \infty_1]_{m_1}, [13, 3, 14; 1]_{m_1}\} \), where all numerical vertex labels are reduced modulo 25. The remaining arcs are isomorphic to \( C_{25}(1, 2, \ldots, 10) \), which has an \( m_1 \)-decomposition by the above comments. The padding is \( \{(13, 3), (14, 13)\} \).

Since \( m_2 \) is the converse of \( m_1 \), the construction of an \( m_2 \)-covering of \( D_v \) will similarly follow.

**Theorem 4.2** A minimum \( d_1 \)-covering of \( D_v \) where \( v \geq 4 \) with padding \( P \) satisfies

(i) \(|A(P)| = 0 \) if \( v \equiv 0 \) or \( 1 \pmod{4} \), and

(ii) \(|A(P)| = 2 \) if \( v \equiv 2 \) or \( 3 \pmod{4} \).

**Minimum \( d_2 \)-coverings of \( D_v \) satisfy the same conditions.**

**Proof.** The necessary conditions follow as in Theorem 4.1. If \( v \equiv 0 \) or \( 1 \pmod{4} \), then there is a decomposition by Theorem 2.2 and the result follows. In the following cases, we have \( V(D_0) = \{\infty_1, \infty_2, 0, 1, \ldots, v - 3\} \).

**Case 1.** Suppose \( v \equiv 2 \pmod{4} \), say \( v = 4k + 2 \) where \( k \geq 2 \). Take the \( d_1 \)-packing of \( D_v \) given in Theorem 3.2 and replace the block \([0, 3, 2; \infty_2]_{d_1}\) with the two blocks \([0, 2, \infty_2; 3]_{d_1}\) and \([2, 3, \infty_1; 6]_{d_1}\). This is a minimum covering of \( D_v \) with padding \( P \) where \( A(P) = \{(2, \infty_2), (3, \infty_1)\} \).

For \( v = 6 \), consider the set of blocks \( \{[5, 0, 1; 4]_{d_1}, [1, 5, 4; 2]_{d_1}, [3, 1, 0; 5]_{d_1}, [2, 4, 3; 1]_{d_1}, [4, 3, 1; 0]_{d_1}, [0, 2, 4; 3]_{d_1}, [5, 2, 3; 4]_{d_1}, [2, 5, 0; 3]_{d_1}\} \). This is a minimum \( d_1 \)-covering of \( D_v \) with padding \( P \) where \( A(P) = \{(3, 2), (4, 5)\} \).

**Case 2.** Suppose \( v \equiv 3 \pmod{4} \), say \( v = 4k + 3 \). Consider the blocks in \( A \cup B \setminus \{[0, 3, \infty_2; 4]_{d_1}\} \cup \{[0, \infty_2, \infty_1; 4]_{d_1}, [\infty_2, 3, 0; \infty_1]_{d_1}\} \) where sets \( A \) and \( B \) are defined in Theorem 3.2 Case 2. This is a minimum covering of \( D_v \) with padding \( P \) where \( A(P) = \{(0, \infty_2), (\infty_1, 0)\} \).

Since \( d_2 \) is the converse of \( d_1 \), the construction of a \( d_2 \)-covering of \( D_v \) will similarly follow.

**Theorem 4.3** A minimum \( d_3 \)-covering of \( D_v \) where \( v \geq 4 \) with padding \( P \) satisfies

(i) \(|A(P)| = 0 \) if \( v \equiv 0 \) or \( 1 \pmod{4} \), and

(ii) \(|A(P)| = 2 \) if \( v \equiv 2 \) or \( 3 \pmod{4} \).

**Minimum \( d_4 \)-coverings of \( D_v \) satisfy the same conditions.**
Proof. The necessary conditions follow as in Theorem 4.2. When $v \equiv 0$ or $1 \pmod{4}$, a decomposition exists by Corollary 2.3 and $|A(P)| = 0$ in this case.

Case 1. Suppose $v \equiv 2 \pmod{8}$, $v \neq 6$. Take the $d_3$-packing of $D_v$ given in Corollary 3.3 and replace the block $[5, 2, 0; 5k + 6]_{d_3}$ with the two blocks $[2, \infty, 0; 5]_{d_3}$ and $[5, \infty, 0; 5k + 6]_{d_3}$. This is a minimum covering of $D_v$ with padding $P$ where $A(P) = \{(2, \infty_2), (5, \infty_3)\}$.

For $v = 6$, take the $d_3$-packing of $D_6$ given in Corollary 3.3, along with the block $[2, 3, 5; 1]_{d_3}$. This yields a minimum covering of $D_6$ with padding $P$ where $A(P) = \{(1, 2), (2, 5)\}$.

Case 2. Suppose $v \equiv 3 \pmod{4}$. Take the $d_3$-packing of $D_v$ given in Corollary 3.3 and replace the block $[0, \infty_1, 4k; 2]_{d_1}$ with the two blocks $[\infty_1, 0, 4k; \infty_2]_{d_3}$ and $[0, \infty_1, \infty_2; 2]_{d_3}$. This is a minimum covering of $D_v$ with padding $P$ where $A(P) = \{(\infty_1, 0), (0, \infty_2)\}$.

Case 3. Suppose $v \equiv 6 \pmod{8}$. Take the $d_3$-packing of $D_v$ given in Corollary 3.3 and replace the block $[5, 2, 0; 5k + 9]_{d_3}$ with the two blocks $[2, \infty_2, 0; 5]_{d_3}$ and $[5, \infty_3, 0; 5k + 9]_{d_3}$. This is a minimum covering of $D_v$ with padding $P$ where $A(P) = \{(2, \infty_2), (5, \infty_3)\}$.

Since $d_4$ is the converse of $d_3$, the construction of a $d_4$-covering of $D_v$ will similarly follow. □

Theorem 4.4 A minimum $d_5$-covering of $D_v$ where $v \geq 4$ with padding $P$ satisfies

(i) $|A(L)| = v$ if $v \equiv 0 \pmod{2}$,

(ii) $|A(L)| = 0$ if $v \equiv 1 \pmod{4}$, and

(iii) $|A(L)| = 2$ if $v \equiv 3 \pmod{4}$.

Minimum $d_v$-coverings of $D_v$ satisfy the same conditions.

Proof. When $v \equiv 1 \pmod{4}$, a decomposition exists by Corollary 2.4 and the result follows. Notice that the vertices of $d_5$ are of in-degrees 0, 0, 2, and 2. So when $v$ is even, a $d_5$-covering of $D_v$ will have a padding $P$ where the in-degree of each vertex of $P$ is odd. So for $v$ even, a $d_5$-covering of $D_v$ with padding $P$ where $|A(P)| = v$ would be minimum (and similarly for a $d_0$-covering of $D_v$). When $v \equiv 3 \pmod{4}$, $|A(D_v)| \equiv 2 \pmod{4}$ and in this case a $d_5$-covering (and similarly for a $d_6$-covering) of $D_v$ with padding $P$ where $|A(P)| = 2$ would be minimum. In the following cases, we have $V(D_v) = \{0, 1, \ldots, v - 1\}$.

Case 1. Suppose $v \equiv 0 \pmod{4}$, say $v = 4k$. Consider the blocks in $A \cup B$ where $A = \{[j, 2k + j, 2k - 1 + j; 4k - 1 + j]_{d_5} | j = 0, 1, \ldots, 4k - 1\}$ and $B = \{[j, k + 1 + i + j, 1 + 2i + j; 4k - 2 - 2i + j]_{d_5} | i = 0, 1, \ldots, k - 2, j =
$0, 1, \ldots, 4k - 1 \}$ where vertex labels are reduced modulo $4k$. Then $A \cup B$ is a minimum $d_5$-covering of $D_v$ with padding $P$ where $A(P) = \{(j, j + 1) \mid j = 0, 1, \ldots, 4k - 1\}$.

**Case 2.** Suppose $v \equiv 2 \pmod{4}$, say $v = 4k + 2$. Consider the blocks in $A \cup B$ where $A = \{[2j, 4k + 1 + 2j, 1 + 2j; 4k + 2j]_{d_5} \mid j = 0, 1, \ldots, 2k\}$ and $B = \{[j, k + 1 + j, 1 + j; 2k + 1 + j]_{d_5} \mid j = 0, 1, \ldots, 4k + 1\} \cup \{[j, k + 2 + i + j, 3 + 2i + j; 4k - 2 - 2i + j]_{d_5} \mid i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 4k + 1\}$ where vertex labels are reduced modulo $4k + 2$. Then $A \cup B$ is a minimum $d_5$-covering of $D_v$ with padding $P$ where $A(P) = \{(j, j + 1) \mid j = 0, 1, \ldots, 4k - 1\}$.

**Case 3.** Suppose $v \equiv 3 \pmod{4}$, say $v = 4k + 3$. Consider the blocks in $A \cup B$ where $A = \{[2j, 4k + 2 + 2j, 1 + 2j; 4k + 1 + 2j]_{d_5} \mid j = 0, 1, \ldots, 2k + 1\}$ and $B = \{[j, 3k - 1 + j, 4k + 2 + j; 4k + j]_{d_5} \mid j = 0, 1, \ldots, 4k + 2\} \cup \{[j, 2k + i + j, 2k + 4 + 2i + j; 2 + 2i + j]_{d_5} \mid i = 0, 1, \ldots, k - 2, j = 0, 1, \ldots, 4k - 1\}$ where vertex labels are reduced modulo $4k + 3$. Then $A \cup B$ is a minimum $d_5$-covering of $D_v$ with padding $P$ where $A(P) = \{(4k + 1, 0), (4k + 2, 0)\}$.

Since $d_6$ is the converse of $d_5$, the construction of a $d_6$-covering of $D_v$ will similarly follow.

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**References**


