AUTOMORPHISMS OF STEINER TRIPLE SYSTEMS

Robert Bentley Gardner, II

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Certificate of Approval:

D. A. Leonard, Associate
Professor, Algebra,
Combinatorics and Analysis

D. G. Hoffman, Chairman
Professor, Algebra,
Combinatorics and Analysis

C. A. Rodger, Associate
Professor, Algebra,
Combinatorics and Analysis

L. M. Teirlinck, Associate
Professor, Algebra,
Combinatorics and Analysis

Norman J. Doorenbos, Dean
Graduate School

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Robert Bentley Gardner, II

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Robert Bentley Gardner, II

Robert Gardner, Jr.
Signature of Author

July 20, 1987
Date

Copy sent to:

Name	Date

VITA

Robert Bentley Gardner, Junior, son of Robert Gardner and Ruby (Gordon) Gardner, was born February 4, 1963, in Montgomery, Alabama. He graduated from Hooper Academy in Hope Hull, Alabama, in 1981. In June, 1981, he entered Auburn University at Montgomery and received the degree of Bachelor of Science (Mathematics) in May, 1984. While an undergraduate, he worked at the Alabama Department of Environmental Management, Air Division. He entered the graduate school of Auburn University in September, 1984.

THESIS ABSTRACT

AUTOMORPHISMS OF STEINER TRIPLE SYSTEMS

Robert Bentley Gardner, II

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Several types of automorphisms of Steiner triple systems are investigated and necessary and sufficient conditions are presented in each case. Cyclic Steiner triple systems, reverse Steiner triple systems, k-rotational Steiner triple systems, and Steiner triple systems with an involution are covered, with examples for each result.

"To the tree from which this book is made."

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I. INTRODUCTION

This thesis is a survey of the existing literature on the automorphisms of Steiner triple systems. The major results are the following:

A cyclic Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$.

A reverse STS(v) exists if and only if $v \equiv 1, 3, 9$ or $19 \pmod{24}$.

A 1-rotational STS(v) exists if and only if $v \equiv 3$ or $9 \pmod{24}$.

A 2-rotational STS(v) exists if and only if $v \equiv 1, 3, 7, 9, 15$, or $19 \pmod{24}$.

A 3-rotational STS(v) exists if and only if $v \equiv 1$ or $19 \pmod{24}$.

A 4-rotational STS(v) exists if and only if $v \equiv 1$ or $9 \pmod{12}$.

A 6-rotational STS(v) exists if and only if $v \equiv 1, 7$, or $19 \pmod{24}$.

There exists a STS(v) on $\{\infty, a, b, 0, 1, \ldots, N-1\}$ where $v = N + 3$, admitting $(\infty)(a, b)(0, 1, \ldots, N-1)$ as an automorphism if and only if $v = 3$ or $v \equiv 1, 7, 9$ or $15 \pmod{24}$, $v > 1$.

There exists a STS(v) admitting an automorphism of type $[f, (v-f)/2, 0, \ldots, 0]$ if and only if $v \equiv 1$ or $3 \pmod 6$, $f \equiv 1$ or $3 \pmod 6$, $f \neq 1$, and $(v-f \equiv 0 \pmod 4$ and $v \geq 2f + 1)$ or $(v-f \equiv 2 \pmod 4$ and $v \not< 3f)$.

And now for a few preliminary results and definitions. A Steiner triple system of order v, denoted STS(v) is a v-element set, X, of points, together with a set, β, of unordered triples of elements of X, called blocks, such that any two points of X are together in exactly one block of β. An automorphism of a STS(v) is a permutation π of X which fixes β. A permutation π of a v element set is said to be of type $[\pi_1, \pi_2, \ldots, \pi_v]$ if the disjoint cyclic decomposition of π contains π_i cycles of length i, and so $\Sigma i\pi_i = v$. The orbit of a block under an automorphism, π, is the image of the block under the various powers of π. A set B of blocks is said to be a set of base blocks for a STS(v) under the permutation π if the orbits of the blocks of B produce the STS(v) and exactly one block of B occurs in each orbit. A difference triple (mod n) is a triple of distinct positive integers where either two of the numbers sum to the third, or the sum of all three is n. An obvious result is the following:

Theorem 1.1. The set of fixed points of an automorphism of STS(v) is the point set of a subsystem of the STS(v).

Another result concerning subsystems is:

Theorem 1.2. (Doyen-Wilson Theorem) Let v,w $\equiv$ 1 or 3(mod 6) with $w \geq 2v + 1$. Then any STS(v) can be embedded in a STS(w).

Also,

Theorem 1.3. Let π be a permutation on a set A. Then there is a set $B \supseteq A$, and a permutation ϕ on B, so that $\phi(a) = \pi(a)$ for all $a \in A$, and there is a Steiner triple system on B admitting ϕ as an automorphism.

Proof: Let $B = Z_2^A - \{0\}$ and identify $a \in A$ with the vector whose a'th entry is one and all other entries are 0. Now to define the permutation ϕ on B, permute the coordinates of the vectors by π. Although A is not a subset of B, the structure of the permutation on A is preserved, so this will do. Now declare (u,v,w) a triple if and only if $u + v + w = 0$. Any pair of vectors $u,v \in B$ are in exactly one triple since $u + v + w = 0$ for exactly one vector w of B ($u+v = 0$ if and only if $u = v$).

II. CYCLIC STEINER TRIPLE SYSTEMS

A STS(v) is said to be <u>cyclic</u> if it admits an automorphism of the type [0, ..., 0,1]. Such a system exists if and only if $v \equiv 1$ or 3 (mod 6) and $v \neq 9$.

In 1897, Heffter [6] made the following observations:

<u>Heffter's first difference problem</u>. The construction of a cyclic STS(v) where $v = 6n + 1$ is equivalent to partitioning the set {1, 2, ..., 3n} into triples such that in each triple the sum of two numbers is equal to the third or the sum of the three is equal to v.

<u>Heffter's second difference problem</u>. The construction of a cyclic STS(v) where $v = 6n + 3$ is equivalent to partitioning the set {1, ..., 2n, 2n+2, ..., 3n+1} into triples with the same properties as above. These triples act as difference triples for the desired systems under the cyclic automorphism.
These problems were first solved by R. Peltesohn [8].

Skolem [12] posed the following restricted version of Heffter's first difference problem: partition the set {1, ..., 2n} into distinct pairs (a_r, b_r) such that $b_r = a_r + r$ for $r = 1, \ldots, n$. Such a partitioning is called an <u>(A,n)-system</u>. If such a partitioning exists, then the triples $(r, a_r + n, b_r + n)$, $r = 1, \ldots, n$ represent a solution to Heffter's first difference problem. Skolem proved the following.

Theorem 2.1. An (A,n)-system exists if and only if $n \equiv 0$ or 1 $(\text{mod } 4)$.

Proof: If there is such a partitioning, then $b_r - a_r = r$ for $r = 1, \ldots, n$ and $\sum b_r - \sum a_r = \frac{1}{2}n\,(n+1)$. Also $\sum a_r + \sum b_r = \frac{2n(2n+1)}{2}$. So $\sum b_r = \frac{n(5n+3)}{4}$ and $n \equiv 0$ or 1 $(\text{mod } 4)$.

Now suppose $n \equiv 0 \pmod 4$ and $n = 4s$. Consider the pairs:

$(4s + r - 1,\ 8s - r + 1)$ $r = 1, \ldots, 2s$

$(r,\ 4s - r - 1)$ $r = 1, \ldots, s - 1$

$(s + r + 1,\ 3s - r)$ $r = 1, \ldots, s - 2$ (omit if $s = 2$)

$(s,\ s + 1)$

$(2s,\ 4s - 1)$

$(2s + 1,\ 6s)$

If $s = 1$, take the pairs $(1,2)$, $(5,7)$, $(3,6)$, and $(4,8)$.

Now suppose $n \equiv 1 \pmod 4$ and $n = 4s + 1$. Consider the pairs:

$(4s + r + 1,\ 8s - r + 3)$ $r = 1, \ldots, 2s$

$(r,\ 4s - r + 1)$ $r = 1, \ldots, s$

$(s + r + 2,\ 3s - r + 1)$ $r = 1, \ldots, s-2$ (omit if $s = 2$)

$(s + 1,\ s + 2)$

$(2s + 1,\ 6s + 2)$

$(2s + 2,\ 4s + 1)$.

If $s = 0$, take the pair (1,2). If $s = 1$, take the pairs (2,3), (8,10), (4,7), (5,9), and (1,6).

These pairs satisfy the conditions of an (A,n)-system.

Example 2.1. Theorem 2.1 produces an (A,13)-system with the following pairs: (4,5), (19,21), (6,9), (18,22), (8,13), (17,23), (3,10), (16,24), (2,11), (15,25), (1,12), (14,26), and (7,20).

Skolem conjectured and O'Keefe [7] proved that the set $\{1, \ldots, 2n-1, 2n+1\}$ could be partitioned into distinct pairs (a_r, b_r) with $b_r = a_r + r$ for $r = 1, \ldots, n$ if and only if $n \equiv 2$ or $3 \pmod 4$. Such a partitioning is called a (B,n)-system.

Theorem 2.2. A (B,n)-system exists if and only if $n \equiv 2$ or 3 (mod 4).

Proof: If there is such a partitioning then $b_r - a_r = r$ for $r = 1, \ldots, n$ and $\Sigma b_r - \Sigma a_r = \frac{1}{2}n(n+1)$. Also $\Sigma a_r + \Sigma b_r = 2n^2 + n+1$. So $\Sigma b_r = \frac{5n^2+3n+2}{4}$ and $n \equiv 2$ or $3 \pmod 4$.

Now suppose $n \equiv 2 \pmod 4$ and $n = 4s + 2$. Consider the pairs:

$(r,\ 4s - r + 2)$ $\quad r = 1, \ldots, 2s$

$(4s + r + 3,\ 8s - r + 4)$ $\quad r = 1, \ldots, s-1$ (omit if $s=1$)

$(5s + r + 2,\ 7s - r + 3)$ $\quad r = 1, \ldots, s-1$ (omit if $s=1$)

$(2s + 1,\ 6s + 2)$

$(4s + 2,\ 6s + 3)$

$(4s + 3,\ 8s + 5)$

$(7s + 3,\ 7s + 4)$

If $s = 0$ take the pairs $(1,2)$ and $(3,5)$.

Now suppose $n \equiv 3 \pmod 4$ and $n = 4s - 1$. Consider the pairs:

$(4s + r,\ 8s - r - 2)$ $\quad r = 1, \ldots, 2s - 2$

$(r,\ 4s - r - 1)$ $\quad r = 1, \ldots, s - 1$

$(s + r + 1,\ 3s - r)$ $\quad r = 1, \ldots, s - 2$ (omit if $s=2$)

$(s,\ s + 1)$

$(2s,\ 4s - 1)$

$(2s + 1,\ 6s - 1)$

$(4s,\ 8s - 1)$

If $s = 1$ take the pairs $(1,2)$, $(3,5)$ and $(4,7)$. These pairs satisfy the conditions of a (B,n)-system.

Example 2.2. Theorem 2.2 produces a $(B,14)$-system with the following pairs: (24,25), (6,8), (19,22), (5,9), (18,23), (4,10), (14,21), (3,11), (17,26), (2,12), (16,27), (1,13), (7,20), and (15,29).

Example 2.3. Theorem 2.2 produces a (B,23)-system with the following pairs: (6,7), (34,36), (11,14), (33,37), (10,15), (32,38), (9,16), (31,39), (8,17), (30,40), (12,23), (29,41), (5,18), (28,42), (4,19), (27,43), (3,20), (26,44), (2,21), (25,45), (1,22), (13,35), and (24,47).

Theorem 2.3. There exists a cyclic STS(v) for all $v \equiv 1 \pmod 6$.

Proof: The triples $(r, a_r + n, b_r + n)$ for $r = 1, \ldots, n$ with the a_r's and b_r's as described in theorems 2.1 and 2.2 are disjoint over $\{1, \ldots, 3n\}$ in Skolem's constructions and disjoint over $\{1,\ldots,3n-1, 3n+1\}$ in O'Keefe's constructions. Hence the triples $(0, r, b_r + n)$ for $r = 1, \ldots, n$ can be considered as the base blocks of a cyclic STS(v) with $v = 6n + 1$.

Example 2.4. Theorem 2.3 produces a cyclic STS(79) based on the (A,13)-system from example 2.1 with the following base blocks: (0,1,18), (0,2,34), (0,3,22), (0,4,35), (0,5,26), (0,6,39), (0,7,23), (0,8,37), (0,9,24), (0,10,38), (0,11,25), (0,12,37), and (0,13,33). The differences between the first two entries of each block cover the set $\{1, 2, \ldots, 13\}$. The differences between the first and third entries cover the set $\{b_r + 13 \mid r = 1, 2, \ldots, 13\}$ and the differences between the second and third entries cover the set $\{a_r + 13 \mid r = 1, 2, \ldots, 13\}$ where the a_r and b_r's are as described in theorem 2.1. Since the union of these three sets is the set $\{1, 2, \ldots, 39\}$, all desired differences are covered. In general, the (A,n)-system or (B,n)-system is used in a similar way.

For Heffter's second difference problem, Rosa [10] presented the constructions.

Theorem 2.4. The set $\{1, \ldots, n, n+2, \ldots, 2n+1\}$ can be partitioned into pairs (a_r, b_r) with $b_r = a_r + r$ for $r = 1, \ldots, n$ if and only if $n \equiv 0$ or $3 \pmod 4$. Such a partitioning is called a (C,n)-system.

Proof: If there is such a partitioning, then $b_r - a_r = r$ for $r = 1, \ldots, n$ and $\Sigma b_r - \Sigma a_r = \frac{1}{2}n(n+1)$. Also $\Sigma a_r + \Sigma b_r = 2n(n+1)$. So $\Sigma b_r = \frac{5n(n+1)}{4}$ and $n \equiv 0$ or $3 \pmod 4$.

Now suppose $n \equiv 0 \pmod 4$ and $n = 4s$. Consider the pairs:

$(r, 4s - r + 1)$	$r = 1, \ldots, s-1$	(omit if $s = 1$)
$(s + r - 1, 3s - r)$	$r = 1, \ldots, s-1$	(omit if $s = 1$)
$(4s + r + 1, 8s - r + 1)$	$r = 1, \ldots, s-1$	(omit if $s = 1$)
$(5s + r + 1, 7s - r + 1)$	$r = 1, \ldots, s-1$	(omit if $s = 1$)

$(2s - 1, 2s)$

$(3s, 5s + 1)$

$(3s + 1, 7s + 1)$

$(6s + 1, 8s + 1)$.

Now suppose $n \equiv 3 \pmod 4$ and $n = 4s - 1$. Consider the pairs:

$(r, 4s - r)$	$r = 1, \ldots, 2s-1$
$(4s + r + 1, 8s - r)$	$r = 1, \ldots, s-2$ (omit if $s = 2$)
$(5s + r, 7s - r - 1)$	$r = 1, \ldots, s-2$ (omit if $s = 2$)
$(2s, 6s - 1)$	
$(5s, 7s + 1)$	
$(4s + 1, 6s)$	
$(7s - 1, 7s)$.	

If $s = 1$ take the pairs (1,2), (5,7), and (3,6).

These pairs satisfy the conditions of a (C,n)-system.

Example 2.5. Theorem 2.4 produces a (C,15)-system with the following pairs: (27,28), (7,9), (22,25), (6,10), (21,26), (5,11), (17,24), (4,12), (20,29), (3,13), (19,30), (2,14), (18,31), (1,15), and (8,23).

Theorem 2.5. The set $\{1, \ldots, n, n+2, \ldots, 2n, 2n+2\}$ can be partitioned into pairs (a_r,b_r) with $b_r = a_r + r$ for $r = 1, \ldots, n$

if and only if $n \equiv 1$ or $2 \pmod 4$, $n \neq 1$. Such a partitioning is called a <u>(D,n)-system</u>.

<u>Proof</u>: If there is such a partitioning, then $b_r - a_r = r$ for $r = 1, \ldots, n$ and $\Sigma b_r - \Sigma a_r = \frac{1}{2}n(n+1)$. Also $\Sigma a_r + \Sigma b_r = 2n^2 + 2n+1$.

So $\Sigma b_r = \dfrac{5n^2 + 5n + 2}{4}$ and $n \equiv 1$ or $2 \pmod 4$.

Now suppose $n \equiv 1 \pmod 4$ and $n = 4s + 1$ where $s \geq 2$. Consider the pairs:

$(r,\ 4s - r + 2)$ $\quad r = 1, \ldots, 2s$

$(5s + r,\ 7s - r + 3)$ $\quad r = 1, \ldots, s$

$(4s + r + 2,\ 8s - r + 3)$ $\quad r = 1, \ldots, s-2$ (omit if $s = 2$)

$(2s + 1,\ 6s + 2)$

$(6s + 1,\ 8s + 4)$

$(7s + 3,\ 7s + 4)$

If $n = 5$ take the pairs $(1,5)$, $(2,7)$, $(3,4)$, $(8,10)$ and $(9,12)$.

Now suppose $n \equiv 2 \pmod 4$ and $n = 4s + 2$ where $s \geq 2$. Consider the pairs:

$(r,\ 4s - r + 3)$ $\quad r = 1, \ldots, 2s$

$(4s + r + 4,\ 8s - r + 4)$ $\quad r = 1, \ldots, s-1$

$(5s + r + 3,\ 7s - r + 3)$ $\quad r = 1, \ldots, s-2$ (omit if $s = 2$)

(2s + 1, 6s + 3)

(2s + 2, 6s + 2)

(4s + 4, 6s + 4)

(7s + 3, 7s + 4)

(8s + 4, 8s + 6)

For $n = 2$ take the pairs (1,2) and (4,6). For $n = 6$ take the pairs (1,6), (2,3), (4,10), (5,9), (8,11) and (12,14). These pairs satisfy the conditions of a (D,n)-system.

Example 2.6. Theorem 2.5 produces a (D,18)-system with the following pairs: (31,32), (36,38), (8,11), (25,29), (7,12), (24,30), (6,13), (20,28), (5,14), (23,33), (4,15), (22,34), (3,16), (21,35), (2,17), (10,26), (1,18), and (9,27).

Theorem 2.6. There exists a cyclic STS(v) for all $v \equiv 3 \pmod 6$, $v \neq 9$.

Proof: The triples $(r, a_r + n, b_r + n)$ for $r = 1, \ldots, n$ with the a_r's and b_r's as described in theorems 2.4 and 2.5 are disjoint over the sets $\{1, \ldots, 2n, 2n+2, \ldots, 3n+1\}$ and $\{1, \ldots, 2n, 2n + 2, \ldots, 3n, 3n + 2\}$ $(n \neq 1)$, respectively. Hence, as in theorem 2.3, the triples $(0, r, b_r + n)$ for $r = 1, \ldots, n$

along with the block $(0, \frac{1}{2}v, \frac{2}{3}v)$ can be considered as base blocks for a cyclic STS(v) with $v = 6n + 3$, $n \neq 1$. There is not a cyclic STS(9) since this would amount to partitioning the set $\{1,2,4\}$ into a difference triple.

So there exists a cyclic STS(v) for all $v \equiv 3 \pmod 6$, $v \neq 9$.

<u>Example 2.7</u>. Theorem 2.6 produces a cyclic STS(93) based on the (C,15)-system from example 2.5 with the following base blocks: (0,1,43), (0,2,24), (0,3,40), (0,4,25), (0,5,41), (0,6,26), (0,7,39), (0,8,27), (0,9,44), (0,10,28), (0,11,45), (0,12,29), (0,13,46), (0,14,30), (0,15,38) and (0,31,62). As in example 2.4, the first 15 base blocks cover the differences in the set $\{1, 2, \ldots, 15\} \cup \{a_r + 15 \mid r = 1, 2, \ldots, 15\} \cup \{b_r + 17 \mid r = 1, 2, \ldots, 15\} = \{1, 2, \ldots, 46\} - \{31\}$. Now, the last block covers the difference 31. So all desired differences are covered.

A simple result that follows from this is the following:

<u>Theorem 2.7</u>. Given an automorphism of type $[0, \ldots, B, \ldots, 0]$ on a set of size v, where $B = v/L$, that is an automorphism consisting of B cycles of length L each, there is a STS(v) on this set admitting this automorphism if and only if $v = L \cdot B \equiv 1$ or $3 \pmod 6$ and $B \neq 1$ when $v = 9$. Such a STS(v) is called <u>semi-regular</u>.

(This is called B-regular in the dissertation of C. J. Cho).

Proof: Clearly the condition is necessary. Now for $v \equiv 1$ or 3 (mod 6) and $v \neq 9$, there exists a cyclic STS(v), that is a STS(v) admitting a cycle α of length v as an automorphism. Now the permutation α^B consists of B disjoint cycles of length L, and this is an automorphism of the STS(v). In the case $L = 3$ and $B = 3$, the blocks (0, 1, 2), (0, 5, 6), (0, 7, 8), and (1, 3, 7) are base blocks for a STS(9) under the automorphism (0, 4, 2) (1, 3, 8) (5, 7, 6).

III. REVERSE STEINER TRIPLE SYSTEMS

A STS(v) is said to be <u>reverse</u> if it admits an automorphism of the type $[1, (v-1)/2, 0, \ldots, 0]$. It will be shown that a reverse STS(v) exists if and only if $v \equiv 1, 3, 9$ or $19 \pmod{24}$. The following five theorems are due to Rosa [11].

<u>Theorem 3.1</u>. If there exists a reverse STS(v) then $v \equiv 1, 3, 9$ or $19 \pmod{24}$.

<u>Proof</u>: Suppose there is a reverse STS(v) on the set $\{\infty, a_1, b_1, \ldots, a_{(v-1)/2}, b_{(v-1)/2}\}$ admitting the automorphism $(\infty)(a_1,b_1) \ldots (a_{(v-1)/2}, b_{(v-1)/2})$. The STS(v) must contain $(v-1)/2$ blocks of the form (∞, a_i, b_i) and no other blocks containing ∞. These blocks are fixed and the remaining $\frac{1}{6}(v-1)(v-3)$ blocks are interchanged in pairs. These blocks are of the forms:

(i) (a_i, a_j, a_k) (iii) (a_i, a_j, b_k)

(ii) (b_i, b_j, b_k) (iv) (a_i, b_j, b_k)

where the number of blocks of forms (i) and (ii) is the same, say X, and the number of blocks of forms (iii) and (iv) is the same, say Y. So $X + Y = \frac{1}{2}\{\frac{1}{6}v(v-1) - \frac{1}{2}(v-1)\}$. There are $\binom{(v-1)/2}{2}$ pairs of a's, each block of form (i) containing three pairs of a's and each block of form (iii) containing one pair of a's. So $3X + Y = \binom{(v-1)/2}{2}$.

Solving these two equations, $X = \frac{1}{48}(v-1)(v-3)$ and $Y = \frac{1}{16}(v-1)(v-3)$. Since $v \equiv 1$ or $3 \pmod 6$, it must be that $v \equiv 1, 3, 9$ or $19 \pmod{24}$.

<u>Theorem 3.2</u>. Let $M = 6m$ and let the sets A, B, C be as follows:

(1) if $m \equiv 0$ or $1 \pmod 4$

$A = \{1, 2, \ldots, M\}$

$B = \{1, 3, 5, \ldots, M-1\}$

$C = \{2, 4, \ldots, M\}$

Notice $A = B \cup C$.

(2) if $m \equiv 2$ or $3 \pmod 4$

$A = \{1, 2, \ldots, M-1, M+1\}$

$B = \{1, 3, 5, \ldots, M-3, M\}$

$C = \{2, 4, 6, \ldots, M-2, M+2\}$

Then there exists a set of triples $\Delta(A,B) = \{(P_r, Q_r, Z_r) \mid Q_r = P_r + Z_r,\ r = 1, \ldots, 3m\}$ such that every element of A appears exactly once in $\{P_r \mid r = 1, \ldots, 3m\} \cup \{Q_r \mid r = 1, \ldots, 3m\}$ and every element of B appears exactly once in $\{Z_r \mid r = 1, \ldots, 3m\}$ and there exists a set of triples $\Delta(C) = \{(u_r, v_r, w_r) \mid w_r = u_r + v_r,\ r = 1, \ldots, m\}$ whose elements cover the whole of C exactly once.

<u>Proof</u>: Define the triples of the set $\Delta(A,B)$ as follows:

(1) $(r, 6m - r + 1, 6m - 2r + 1)$, $r = 1, \ldots, 3m$ if $m \equiv 0$ or $1 \pmod 4$

(2) $(1, 6m + 1, 6m)$, $(r, 6m - r + 1, 6m - 2r + 1)$, $r = 2, 3, \ldots, 3m$ if $m \equiv 2$ or $3 \pmod 4$.

Also define the triples of the set $\Delta(C)$ as $(2r, 2p_r + 2m, 2q_r + 2m)$,

$r = 1, 2, \ldots, m$ where (p_r, q_r) are pairs of an (A,m)-system or (B,m)-system according to whether $m \equiv 0$ or 1 (mod 4) or $m \equiv 2$ or 3 (mod 4). These sets meet the requirements of the sets $\Delta(A,B)$ and $\Delta(C)$.

Example 3.1. Suppose $m = 4$, $M = 24$ and $A = \{1, 2, \ldots, 24\}$, $B = \{1, 3, 5, \ldots, 21, 23\}$, and $C = \{2, 4, 6, \ldots, 22, 24\}$. Then theorem 3.2 yields the following collections of triples: $\Delta(A,B) =$ (1, 24, 23), (2, 23, 21), (3, 22, 19), (4, 21, 17), (5, 20, 15), (6, 19, 13), (7, 18, 11), (8,7, 9), (9, 16, 7), (10, 15, 5), (11, 14, 3), and (12, 13, 1); $\Delta(C) =$ (2, 10, 12), (4, 18, 22), (6, 14, 20), and (8, 16, 24). $\Delta(C)$ is based on the (A,4) - system listed in chapter 2. Since this system covers the set {1, 2, 3, 4, 5, 6, 7, 8}, the triples of $\Delta(C)$ cover the set C. Also, since $q_r = p_r + r$ for the pairs in the (A,4)-system, the third element of each triple will be the sum of the other two.

Theorem 3.3. Let $M = 6m + 2$, and let the sets A, B, C, $A = B \cup C$ be as follows:

$$A = \{1, 2, \ldots, 6m + 2\} - \{3m + 1, 6m + 2\}$$

$$B = \{1, 2, \ldots, 2m, 3m + 2, 3m + 4, \ldots, 5m - 2, 5m\}$$

$$C = \{2m + 1, 2m + 2, \ldots, 3m, 3m + 3, 3m + 5, \ldots, 5m - 1, 5m + 1, 5m + 2, \ldots, 6m + 1\}.$$

Then there exists a set of triples $\Delta(A,B) = \{(P_r, Q_r, Z_r) \mid Q_r = P_r + Z_r, r = 1, \ldots, 3m\}$ such that every element of A appears exactly once in $\{P_r \mid r = 1, \ldots, 3m\} \cup \{Q_r \mid r = 1, \ldots, 3m\}$ and every element of B appears exactly once in $\{Z_r \mid r = 1, \ldots, 3m\}$, and there exists a set of triples $\Delta(C) = \{(u_r, v_r, w_r) \mid u_r + v_r + w_r = 2M,\ r = 1, \ldots, m\}$. such that every element of C is covered exactly once by $\{u_r \mid r = 1, \ldots, m\} \cup \{v_r \mid r = 1, \ldots, m\} \cup \{w_r \mid r = 1, \ldots, m\}$.

<u>Proof</u>:

Construct the set $\Delta(A,B)$ as follows:

(1) if m is even, say $m = 2k$, then take

$(k + 1 + r,\ 11k + 1 - r,\ 10k - 2r)$ $r = 0, 1, \ldots, 2k-1$

$(r + 1,\ 4k - r,\ 4k - 1 - 2r)$ $r = 0, 1, \ldots, k-1$

$(4k + 1 + r,\ 6k - r,\ 2k - 1 - 2r)$ $r = 0, 1, \ldots, k-1$

$(8k + 1 + r,\ 12k + 1 - r,\ 4k - 2r)$ $r = 0, 1, \ldots, k-1$

$(7k + 1,\ 9k + 1,\ 2k)$

$(6k + 2 + r,\ 8k - r,\ 2k - 2 - 2r)$ $r = 0, 1, \ldots, k-2$ (omit if $k = 1$).

(2) if m is odd, say $m = 2k + 1$, then take

$(k + 2 + r,\ 11k + 7 - r,\ 10k + 5 - 2r)$ $r = 0, 1, \ldots, 2k$

$(r + 1,\ 4k + 3 - r,\ 4k + 2 - 2r)$ $r = 0, 1, \ldots, k$

$(6k + 5 + r,\ 8k + 5 - r,\ 2k - 2r)$ $r = 0, 1, \ldots, k-1$ (omit if $k = 0$)

$(4k + 4 + r,\ 6k + 3 - r,\ 2k - 1 - 2r)$ $r = 0, 1, \ldots, k-1$ (omit if $k = 0$)

$(8k + 6 + r,\ 12k + 7 - r,\ 4k + 1 - 2r)$ $r = 0, 1, \ldots, k-1$ (omit if $k = 0$)

$(7k + 5,\ 9k + 6,\ 2k + 1)$

Now for $\Delta(C)$ take

$$(2m + r,\ 5m + 3 - 2r,\ 5m + 1 + r) \qquad r = 1, 2, \ldots, m.$$

These triples satisfy the necessary conditions.

Example 3.2. Suppose $m = 4$, $M = 26$ and $A = \{1, 2, \ldots, 12, 14, 15, \ldots, 25\}$, $B = \{1, 2, \ldots, 8, 14, 16, 18, 20\}$, and $C = \{9, 10, 11, 12, 15, 17, 19, 21, 22, 23, 24, 25\}$. Then theorem 3.3 yields the following collections of triples: $\Delta(A,B) = (3, 23, 20), (4, 22, 18), (5, 21, 16), (6, 20, 14), (17, 25, 8); (18, 24, 6), (1, 8, 7), (2, 7, 5), (9, 12, 3), (10, 11, 1), (15, 19, 4)$, and $(14, 16, 2)$; $\Delta(C) = (9, 21, 22), (10, 19, 23), (11, 17, 24)$ and $(12, 15, 25)$.

Theorem 3.4. Let $M = 6m - 1$, $m \geq 2$ and let the sets A, B, C be as follows:

$$A = \{1, 2, \ldots, 6m - 1\} - \{3m\}$$

$$B = \{1, 3, \ldots, 2m - 1, 2m + 1, 2m + 2, \ldots, 3m - 1, 3m + 1, 3m + 2, \ldots, 4m - 1, 5m\}$$

$$C = \{2, 4, \ldots, 2m - 2, 4m + 1, 4m + 2, \ldots, 5m - 1, 5m + 1, 5m + 2, \ldots, 6m - 1\}$$

$$|A| = 6m - 2, \quad |B| = 3m-1, \quad |C| = 3m - 3,$$

$$A = B \cup C \cup \{2m, 4m\}.$$

Then there exists a set of triples $\Delta(A,B) = \{(P_r, Q_r, Z_r) | Q_r = P_r + Z_r, r = 1, \ldots, 3m-1\}$ such that every element of A appears exactly once in $\{P_r | r = 1, \ldots, 3m - 1\} \cup \{Q_r | r = 1, \ldots, 3m-1\}$ and every element of B appears exactly once in $\{Z_r \mid r = 1, \ldots, 3m-1\}$ and there exists a set of triples $\Delta(C) = \{(u_r, v_r, w_r) \mid w_r = u_r + v_r, r = 1, \ldots, m-1\}$ such that every element of C is covered exactly once by $\{u_r \mid r = 1, \ldots, m-1\} \cup \{v_r \mid r = 1, \ldots, m-1\} \cup \{w_r \mid r = 1, \ldots, m-1\}$.

Proof: Construct the set $\Delta(A,B)$ as follows:

(1) if m is even, say $m = 2k$, then take

$(1, 10k + 1, 10k)$

$(r+2, 6k-1-r, 6k-3-2r)$ $r = 0, 1, \ldots, 2k-2$

$(2k+1+r, 10k-r, 8k-1-2r)$ $r = 0, 1, \ldots, k$

$(7k+r, 9k-1-r, 2k-1-2r)$ $r = 0, 1, \ldots, k-1$

$(3k+2+r, 11k-r, 8k-2-2r)$ $r = 0, 1, \ldots, k-2$ (omit if $k = 1$)

$(6k+1+r, 12k-1-r, 6k-2-2r)$ $r = 0, 1, \ldots, k-2$ (omit if $k = 1$).

(2) if m is odd, say $m = 2k+1$, $k \geq 1$, then take

$(1, 10k+6, 10k+5)$

$(8k+3, 8k+4, 1)$

$(6k+4+r, 12k+5-r, 6k+1-2r)$ $r = 0, 1, \ldots, 2k-2$

$(k+1+r, 9k+4-r, 8k+3-2r)$ $r = 0, 1, \ldots, k-1$

$(2k+3+r, 10k+5-r, 8k+2-2r)$ $r = 0, 1, \ldots k$

$(2k+1+r, 4k+4-r, 2k+3-2r)$ $r = 0, 1$

$(2+r, 6k+2-r, 6k-2r)$ $r = 0,1, \ldots, k-2$ (omit if $k = 1$)

$(3k+4+r, 5k+3-r, 2k-1-2r)$ $r = 0, 1, \ldots, k-2$ (omit if $k = 1$).

Now for $\Delta(C)$ take

$(2r, 5m-r, 5m+r)$ $r = 1, 2, \ldots, m - 1$.

These triples satisfy the necessary conditions.

Example 3.3. Suppose $m = 4$, $M = 23$, and $A = \{1, 2, \ldots, 11, 13, 14, \ldots, 23\}$, $B = \{1, 3, 5, 7, 9, 10, 11, 13, 14, 15, 20\}$, and $C = \{2, 4, 6, 17, 18, 19, 21, 22, 23\}$. Then theorem 3.4 yields the following collection of triples: (1, 21, 20), (2, 11, 9), (3, 10, 7), (4, 9, 5), (5, 20, 15), (16, 19, 13), (7, 18, 11), (14, 17, 3), (15, 16, 1), (8, 22, 14), and (13, 23, 10); $\Delta(C)$ = (2, 19, 21), (4, 18, 22), and (6, 17, 23).

Theorem 3.5. Let $v \equiv 1$, 3 or 9 (mod 24), $v \neq 25$. Then there exists a reverse STS(v).

Proof:

Clearly the theorem is true for $v = 3$. For $v = 9$, consider the STS(9) with blocks

$$\begin{array}{lll} (\infty, a_1, b_1) & (a_1, a_2, a_3) & (a_2, a_4, b_1) \\ (\infty, a_2, b_2) & (b_1, b_2, b_3) & (b_2, b_4, a_1) \\ (\infty, a_3, b_3) & (a_1, a_4, b_3) & (a_3, a_4, b_2) \\ (\infty, a_4, b_4) & (b_1, b_4, a_3) & (b_3, b_4, a_2). \end{array}$$

This admits $(\infty)(a_1,b_1)(a_2,b_2)(a_3,b_3)(a_4,b_4)$ as an automorphism. So the theorem holds for $v = 9$.

Now let $N = (v-1)/2$ and let the elements of the system be ∞, $a_1, \ldots, a_N, b_1, \ldots, b_N$ so that the automorphism will be $(\infty)(a_1,b_1) \ldots (a_N, b_N)$.

(1) Suppose $v \equiv 3$ (mod 24), say $v = 24m + 3$ and so $N = 12m + 1$. Consider blocks of the form (∞, a_i, b_i) $i = 1, \ldots, N$.

These are all the blocks containing ∞. Now consider the blocks

$$(a_i,\ a_{i+Z_r},\ b_{i+Q_r}) \quad i = 1, 2, \ldots, N$$

$$(b_i,\ b_{i+Z_r},\ a_{i+Q_r}) \quad i = 1, 2, \ldots, N$$

where the subscripts are reduced modulo N, except "N" is used instead of "0", and (P_r, Q_r, Z_r) run over all triples of $\Delta(A,B)$ as described in theorem 3.2.

Now add the blocks

$$(a_i,\ a_{i+u_r},\ a_{i+w_r}) \quad i = 1, 2, \ldots, N$$

$$(b_i,\ b_{i+u_r},\ b_{i+w_r}) \quad i = 1, 2, \ldots, N$$

where (u_r, v_r, w_r) run over all triples of $\Delta(C)$ as described in theorem 3.2.

(2) Suppose $v \equiv 9 \pmod{24}$, say $v = 24m + 9$ and so $N = 12m + 4$.

As in case 1, take the blocks of the form (∞, a_i, b_i) $i = 1, 2, \ldots, N$, and also take

$$(a_i,\ a_{i+Z_r},\ b_{i+Q_r}) \quad i = 1, 2, \ldots, N$$

$$(b_i,\ b_{i+Z_r},\ a_{i+Q_r}) \quad i = 1, 2, \ldots, N$$

where (P_r, Q_r, Z_r) run over all triples of $\Delta(A,B)$ as described in theorem 3.3. Add the blocks

$$(a_i,\ a_{i+u_r},\ a_{i+u_r+v_r}) \quad i = 1, 2, \ldots, N$$

$$(b_i,\ b_{i+u_r},\ b_{i+u_r+v_r}) \quad i = 1, 2, \ldots, N$$

where (u_r, v_r, w_r) run over all triples of $\Delta(C)$ as described in theorem 3.3. Now add the blocks:

$$(a_i, a_{i+9m+3}, b_{i+6m+2})$$
$$(a_{i+3m+1}, a_{i+9m+3}, b_i)$$
$$(a_{i+6m+2}, a_{i+9m+3}, b_{i+3m+1})$$
$$(b_i, b_{i+9m+3}, a_{i+6m+2})$$
$$(b_{i+3m+1}, b_{i+9m+3}, a_i)$$
$$(b_{i+6m+2}, b_{i+9m+3}, a_{i+3m+1})$$
$$(a_i, a_{i+3m+1}, a_{i+6m+2})$$
$$(b_i, b_{i+3m+1}, b_{i+6m+2})$$

where i ranges from 1 to $3m+1$.

(3) Suppose $v \equiv 1 \pmod{24}$, say $v = 24m+1$ $(m \neq 1)$ and so $N = 12m$ $(m \geq 2)$.

Take the same blocks as described in case (1) but use the $\Delta(A,B)$ and $\Delta(C)$ as described in theorem 3.4. Now add the blocks

$$(a_i, a_{i+9m}, b_{i+6m}), (b_i, b_{i+9m}, a_{i+6m}),$$
$$(a_{i+3m}, a_{i+9m}, b_i), (b_{i+3m}, b_{i+9m}, a_i)$$
$$(a_{i+6m}, a_{i+9m}, b_{i+3m}), (b_{i+6m}, b_{i+9m}, a_{i+3m}),$$
$$(a_i, a_{i+3m}, a_{i+6m}), (b_i, b_{i+3m}, b_{i+6m})$$

for $i = 1, 2, \ldots, 3m$ and

$$(a_i, a_{i+2m}, a_{i+4m}), (b_i, b_{i+2m}, b_{i+4m}),$$
$$(a_{i+4m}, a_{i+6m}, a_{i+8m}), (b_{i+4m}, b_{i+6m}, b_{i+8m}),$$
$$(a_{i+8m}, a_{i+10m}, a_i), (b_{i+8m}, b_{i+10m}, b_i)$$
$$(a_{i+2m}, a_{i+6m}, a_{i+10m}), (b_{i+2m}, b_{i+6m}, b_{i+10m})$$

for $i = 1, 2, \ldots, 2m$.

These blocks satisfy the conditions for a STS(v) and clearly admit the desired automorphism.

Example 3.4. Suppose $v = 99$, $m = 4$ and $N = 49$. Then, by case 1 of theorem 3.5, a reverse STS(99) can be constructed on the set $\{\infty, a_1, a_2, \ldots, a_{49}, b_1, b_2, \ldots, b_{49}\}$ admitting the automorphism $\alpha = (\infty)(a_1,b_1)(a_2,b_2) \ldots (a_{49}, b_{49})$. Define x as a pure difference (mod N) if it is the difference (mod N) associated with the subscripts of a pair of elements of type (a_i, a_j) or (b_i, b_j). Define x as a mixed difference (mod N) if it is the difference (mod N) associated with the subscripts of a pair of elements of type (a_i, b_j). So the mixed differences should cover the set $\{0, 1, 2, \ldots, (N-1)/2\}$ and the pure differences should cover the set $\{1, 2, \ldots, (N-1)/2\}$. First, take the blocks (∞, a_i, b_i) for $i = 1, 2, \ldots, 52$. This takes care of all blocks containing ∞ and all blocks containing a pair of elements with mixed difference 0. Using $\Delta(A,B)$ as described in example 3.1, we get the following blocks, along with their images under α: (a_i, a_{i+23}, b_i), $(a_i, a_{i+21}, b_{i+23})$, $(a_i, a_{i+19}, b_{i+23})$, $(a_i, a_{i+17}, b_{i+22})$, $(a_i, a_{i+15}, b_{i+20})$, $(a_i, a_{i+14}, b_{i+19})$, $(a_i, a_{i+11}, b_{i+18})$, (a_i, a_{i+9}, b_{i+17}), (a_i, a_{i+7}, b_{i+16}), (a_i, a_{i+6}, b_{i+15}), (a_i, a_{i+3}, b_{i+14}), and (a_i, a_{i+2}, b_{i+13}) for $i = 1, 2, \ldots, 49$. These cover all pure differences in the set B described in example 3.1, that is the set $\{1, 3, 5, \ldots, 23\}$. They also cover all mixed differences in the set $A = \{1, 2, \ldots, 24\}$, from example 3.1. Now, using $\Delta(C)$ as described in theorem 3.2, we get the blocks:

(a_i, a_{i+2}, a_{i+13}), (a_i, a_{i+5}, a_{i+22}), (a_i, a_{i+6}, a_{i+21}), and (a_i, a_{i+8}, a_{i+24}) for $i = 1, 2, \ldots, 49$. These cover all pure differences in the set $C = \{2, 4, 6, \ldots, 24\}$ from example 3.1. So all desired mixed and pure differences are covered, and these are the blocks of a reverse STS(99). Similarly, in general, the mixed differences in the set $A \cup \{0\}$ and the pure differences in the sets B and C are covered, and these are all the desired differences.

Example 3.5. Suppose $v = 105$, $m = 4$ and $N = 52$. Then, by case 2 of theorem 3.5, a reverse STS(105) can be constructed on the set $\{\infty, a_1, a_2, \ldots, a_{52}, b_1, b_2, \ldots, b_{52}\}$ admitting the automorphism $\alpha = (\infty)(a_1,b_1)(a_2,b_2) \ldots (a_{52}, b_{52})$. First take the blocks (∞, a_i, b_i) for $i = 1, 2, \ldots, 52$. This takes care of all blocks containing ∞ and all blocks containing a pair of elements with mixed difference 0. Using $\Delta(A,B)$ as described in example 3.2, we get the following blocks, along with their images under α: $(a_i, a_{i+20}, b_{i+24})$, $(a_i, a_{i+19}, b_{i+22})$, $(a_i, a_{i+16}, b_{i+22})$, $(a_i, a_{i+14}, b_{i+20})$, (a_i, a_{i+8}, b_{i+25}), (a_i, a_{i+6}, b_{i+24}), (a_i, a_{i+7}, b_{i+8}), (a_i, a_{i+5}, b_{i+7}), (a_i, a_{i+3}, b_{i+12}), (a_i, a_{i+1}, b_{i+11}), (a_i, a_{i+4}, b_{i+19}), and (a_i, a_{i+2}, b_{i+16}) for $i = 1, 2, \ldots, 52$. These cover all pure differences in the set $B = \{1, 2, \ldots, 8, 14, 16, 18, 20\}$ from example 3.2. They also cover all mixed differences in the set $A = \{1, 2, \ldots, 12, 14, 15, \ldots, 25\}$ from example 3.2. Now using $\Delta(C)$ as described in example 3.2, we get the following blocks, along with their images under α: (a_i, a_{i+9}, a_{i+30}), $(a_i, a_{i+10}, a_{i+29})$, $(a_i, a_{i+11}, a_{i+28})$, and $(a_i, a_{i+12}, a_{i+27})$ for $i = 1, 2, \ldots, 52$. These cover all pure differences in the set $C = \{9, 10, 11, 12, 15, 17, 21, 22, 23, 24, 25\}$ from example 3.2. So all pure and mixed differences except 13 and 26 have been covered. Now, add the following blocks, along with their images under α: $(a_i, a_{i+39}, b_{i+26})$, $(a_{i+13}, a_{i+39}, b_i)$, $(a_{i+26}, a_{i+39}, b_{i+13})$, and $(a_i, a_{i+13}, a_{i+26})$ for $i = 1, 2, \ldots, 13$. These cover the missing differences, and so these are the blocks of a reverse STS(105). Similarly, in general, the blocks based on $\Delta(A,B)$ and $\Delta(C)$ cover all mixed differences in the set $A \cup \{0\}$ and all pure differences in the sets B and C. These are all the desired differences except $3m + 1$ and $6m + 2$ which are covered by the addition of the other blocks.

<u>Example 3.6</u>. Suppose $v = 47$, $m = 4$ and $N = 48$. Then, by case 3 of theorem 3.5, a reverse STS(97) can be constructed on the set $\{\infty, a_1, a_2, \ldots, a_{48}, b_1, b_2, \ldots, b_{48})$ admitting the automorphism $\alpha = (\infty)(a_1,b_1)(a_2,b_2) \ldots (a_{48}, b_{48})$. First, take the blocks (∞, a_i, b_i) for $i = 1, 2, \ldots, 48$. This takes care of all blocks containing ∞ and all blocks containing a pair of elements with mixed difference 0. Using $\Delta(A,B)$ as described in example 3.3, we get the following blocks, along with their images under α: $(a_i, a_{i+20}, b_{i+21})$, (a_i, a_{i+9}, b_{i+11}), (a_i, a_{i+7}, b_{i+10}), (a_i, a_{i+5}, b_{i+10}), $(a_i, a_{i+15}, b_{i+20})$, $(a_i, a_{i+13}, b_{i+19})$, $(a_i, a_{i+11}, b_{i+18})$, (a_i, a_{i+3}, b_{i+17}),

(a_i, a_{i+1}, b_{i+16}), $(a_i, a_{i+14}, b_{i+22})$, and $(a_i, a_{i+10}, b_{i+23})$ for $i = 1, 2, \ldots, 48$. These cover all pure differences in the set $B = \{1, 3, 5, 7, 9, 10, 11, 13, 14, 15, 20\}$ from example 3.3. They also cover all mixed differences in the set $A = \{1, 2, \ldots, 11, 13, 14, \ldots, 23\}$ from example 3.3. Now using $\Delta(C)$ as described in example 3.3, we get the following blocks, along with their images under α:
(a_i, a_{i+2}, a_{i+22}), (a_i, a_{i+4}, a_{i+22}), and (a_i, a_{i+6}, a_{i+23}) for $i = 1, 2, \ldots, 48$. These cover all pure differences in the set $C = \{2, 4, 6, 17, 18, 19, 21, 22, 23\}$ from example 3.3. So all mixed differences are covered except 12 and 24. All pure differences are covered except 8, 12, 16, and 24. The following blocks cover these differences: $(a_i, a_{i+36}, b_{i+24})$, $(a_{i+12}, a_{i+36}, b_i)$, $(a_{i+24}, a_{i+36}, b_i)$ and $(a_i, a_{i+12}, a_{i+24})$ for $i = 1, 2, \ldots, 12$; and (a_i, a_{i+8}, a_{i+16}), $(a_{i+16}, a_{i+24}, a_{i+32})$, $(a_{i+32}, a_{i+40}, a_i)$ and $(a_{i+8}, a_{i+24}, a_{i+40})$ for $i = 1, 2, \ldots, 8$. Similarly, in general, the blocks based on $\Delta(A,B)$ and $\Delta(C)$ cover all mixed differences in the set $A \cup \{0\}$ and all pure differences in sets B and C. These are all the desired mixed differences except 3m and 6m and all the desired pure differences except 2m, 3m, 4m and 6m. These differences are covered by the addition of the other blocks.

In the same paper, Rosa also showed the existence of a reverse STS(19). Consider:

(∞, a_i, b_i) $i = 1, 2, \ldots, 9$

(a_1, a_4, a_7), (a_2, a_5, a_8), (a_3, a_6, a_9), (a_1, a_5, a_9),

(a_2, a_6, a_7), (a_3, a_4, a_8), (a_1, a_2, b_6), (a_1, a_3, b_5),

$(a_2, a_3, b_4), (a_4, a_5, b_9), (a_4, a_6, b_8), (a_5, a_6, b_7),$

$(a_7, a_8, b_3), (a_7, a_9, b_2), (a_8, a_9, b_1), (a_1, a_6, b_3),$

$(a_1, a_8, b_7), (a_6, a_8, b_5), (a_2, a_4, b_1), (a_2, a_9, b_8),$

$(a_4, a_9, b_6), (a_3, a_5, b_2), (a_3, a_7, b_9), (a_5, a_7, b_4)$

along with the images of these blocks under $\alpha = (\infty)(a_1,b_1) \ldots (a_9,b_9)$. These are the blocks of a STS(19) admitting α as a permutation.

Doyen presented a reverse STS(25) [4]. Unfortunately there were typographical errors in the paper. It was similar to the following reverse STS(25):

$(\infty, a_i, b_i) \quad i = 1, \ldots, 12$

$(a_1, a_3, b_4), (a_2, a_3, a_4), (a_1, a_2, b_3), (a_2, b_1, b_4),$

$(a_5, a_7, b_8), (a_6, a_7, a_8), (a_5, a_6, b_7), (a_5, a_8, b_6),$

$(a_9, a_{11}, b_{12}), (a_{10}, a_{11}, a_{12}), (a_9, a_{10}, b_{11}), (a_9, a_{12}, b_{10}),$

$(a_1, a_5, a_9), (a_2, a_5, a_{12}), (a_3, a_5, b_{12}), (a_4, a_5, b_9),$

$(a_1, a_6, a_{11}), (a_2, a_6, a_{10}), (a_3, a_6, b_{11}), (a_4, a_6, a_{10}),$

$(a_1, a_7, a_{10}), (a_2, a_7, b_{10}), (a_3, a_7, a_{11}), (a_4, a_7, b_{11}),$

$(a_1, a_8, b_9), (a_2, a_8, b_{12}), (a_3, a_8, a_9), (a_4, a_8, a_{12})$

$(a_1, b_5, b_{10}), (a_2, b_5, b_{11}), (a_3, b_5, a_{10}), (a_4, b_5, b_{11})$

$(a_1, b_6, a_{12}), (a_2, b_6, a_9), (a_3, b_6, b_9), (a_4, b_6, b_{12})$

$(a_1, b_7, b_{12}), (a_2, b_7, b_9), (a_3, b_7, a_{12}), (a_4, b_7, a_9)$

$(a_1, b_8, b_{11}), (a_2, b_8, a_{11}), (a_3, b_8, b_{10}), (a_4, b_8, a_{10})$

along with the images of these blocks under $\alpha = (\infty)(a_1,b_1)\ldots(a_{12},b_{12})$. These are the blocks of a STS(25) admitting α as a permutation.

This system contains three subsystems of order 9, namely $\{\infty, 1, 2, 3, 4, 1', 2', 3', 4'\}$, $\{\infty, 5, 6, 7, 8, 5', 6', 7', 8'\}$ and $\{\infty, 9, 10, 11, 12, 9', 10', 11', 12'\}$. Teirlinck [14] completed the problem with the following two theorems.

Theorem 3.6. The set $\{2, 3, \ldots, 2n-1, 2n, 2n+2, 2n+4, 2n+5, \ldots, 8n+1, 8n+2, 8n+4\}$ can be partitioned into $4n$ pairs (p_r,q_r) such that $q_r - p_r = r$ for $r = 1, \ldots, 4n$.

Proof: For $n > 1$, consider the pairs:

$(2+i,\ 4n+2-i)$ $i = 0, \ldots, 2n-2$,

$(4n+4+i,\ 8n+1-i)$ $i = 0, \ldots, n-2$,

$(5n+2+i,\ 7n+1-i)$ $i = 1, \ldots, n-2$ (omit if $n = 2$),

$(2n+2,\ 6n+1)$,

$(4n+3,\ 6n+2)$,

$(7n+1,\ 7n+2)$,

$(8n+2,\ 8n+4)$.

For $n = 1$ consider $(2,6)$, $(4,7)$, $(10,12)$, $(8,9)$.

These pairs satisfy the necessary conditions.

Example 3.7. Suppose $n = 4$, and consider the set $\{2, 3, 4, 5, 6, 8, 10, 11, \ldots, 25, 26, 28\}$. Then theorem 3.6 yields the following pairs: (22,23), (26,28), (18, 21), (6,10), (15,20), (5,11), (17,24), (4,12), (16,25), (3,13), (8,19) and (2,14).

<u>Theorem 3.7</u>. There exists a reverse STS(v) for every $v \equiv 19 \pmod{24}$, $v > 19$.

<u>Proof</u>: Let $v = 24n + 19$ $(n > 0)$ and choose a reverse STS(9) on the elements $\{\infty, a, a', b, b', c, c', d, d'\}$ admitting the automorphism $(\infty)(a,a')(b,b')(c,c')(d,d')$. Now consider the 12 blocks of this STS(9) along with the images of the blocks $(\infty, 0, 12n+5)$, $(a, 0, 4n+1)$, $(b, 0, 6n+1)$, $(c, 0, 6n + 3)$, $(d, 0, 12n + 3)$, and $(0, r, 4n + p_r + r)$ for $r = 1, \ldots, 4n$ where p_r is as described in theorem 3.6, under the various powers of the permutation $\omega = (\infty)(a,a')(b,b')(c,c')(d,d')(0, 1, \ldots, 24n + 9)$. These are the blocks for a STS(24n + 19) which admits ω as an automorphism. Now the permutation $\alpha = \omega^{12n+5}$ is a permutation of type $[1, 12n + 9, 0, \ldots, 0]$ and the STS (24n + 19) admits α as an automorphism. So, the theorem follows.

<u>Example 3.8</u>. Suppose $n = 3$ and $v = 91$. Consider the reverse STS(9) with blocks: (∞, a, a'), (∞, b, b'), (∞, c, c'), (∞, d, d'), (a, d, c'), (a', d', c), (b, d, a'), (b', d', a), (c, d, b'), (c', d', b), (a, b, c) and (a', b', c'). These blocks include all possible pairs of the forms (x,y), (x',y) or (x',y') where $x, y \in \{a, b, c, d\}$. Now add the triples $(\infty, 0, 41)$, $(a, 0, 13)$, $(b, 0, 19)$, $(c, 0, 1)$, and $(d, 0, 39)$ and their images under $\omega = (\infty)(a,a')(b,b')(c,c')(d,d')$

(0, 1, ..., 81). These triples contain all pairs of type (∞,x), (a,x), (a',x), (b,x), (b',x), (c,x), (c',x), (d,x), and (d',x) where $x \in Z_{82}$, and all pairs of numbers with differences 13, 19, 21, 39, and 41. Now based on the number pairs in example 3.7, add the triples (0, 1, 35), (0, 2, 40), (0, 3, 33), (0, 4, 22), (0, 5, 32), (0, 6, 23), (0, 7, 36), (0, 8, 24), (0, 9, 37), (0, 10, 25), (0, 11, 31), and (0, 12, 26) and their images under ω. These triples include all number pairs with differences in the set $\{r, p_r + 12, q_r + 12 \mid r=1,2, \ldots,12\}$ = {1, 2, ..., 12, 14, 15, 16, 17, 18, 20, 22, 23, ..., 37, 38, 40}. So all desired pairs are present and this is a STS(91) on the set $\{\infty$, a, a', b, b', c, c', d, d', 0, 1, ..., 81}. Similarly, in general, the blocks based on the pairs of theorem 3.6 cover all differences in the set {1, 2, ..., 4n, 4n + 2, 4n + 3, ..., 6n−1, 6n, 6n+2, 6n+4, 6n+5, ..., 12n+1, 12n+2, 12n+4}, and the other blocks cover the remaining differences and desired pairs.

So, in conclusion a reverse STS(v) exists if and only if $v \equiv 1$, 3, 9 or 19 (mod 24).

IV. k-ROTATIONAL STEINER TRIPLE SYSTEMS

A STS(v) is said to be k-rotational if it admits an automorphism of type [1, 0, ..., 0, k, 0, ..., 0], that is, an automorphism with one fixed point and k cycles of length $(v-1)/k$ each. Necessary and sufficient conditions exist for $k = 1, 2, 3, 4$ and 6.

Phelps and Rosa [9] presented the results for $k = 1, 2$, and 6.

Theorem 4.1. If there exists a 1-rotational STS(v) then $v \equiv 3$ or 9 (mod 24).

Proof: Let $X = Z_{v-1} \cup \{\infty\}$ and let $\alpha = (\infty)(0, 1, \ldots, v-2)$ be an automorphism of a 1-rotational STS(v) with point set X. Now, there are $\frac{1}{2}(v-1)$ blocks containing ∞, each of the form $(\infty, i, i + \frac{1}{2}(v-1))$.

All blocks of the STS(v) not containing ∞ are partitioned into orbits under α of length $v-1$ except possibly a short orbit Q_0 of length $\frac{1}{3}(v-1)$ which contains the block $(0, \frac{1}{3}(v-1), \frac{2}{3}(v-1))$.

No 1-rotational STS(v) contains blocks of Q_0 since this would require $v \equiv 1 \pmod 6$, and also, the remaining $\frac{1}{6}(v-1)(v-5)$ blocks not in Q_0 or containing ∞ must be partitioned into $\frac{1}{6}(v-5)$ orbits of length $v-1$. But if $v \equiv 1 \pmod 6$ then $6 \nmid (v-5)$.

So the $\frac{1}{6}(v-1)(v-3)$ blocks not containing ∞ must be partitioned into $\frac{1}{6}(v-3)$ orbits of length $v-1$. So $v \equiv 3 \pmod 6$.

Since $v-1$ is even, the automorphism $\alpha^{(v-1)/2}$ is a permutation of type $[1, \frac{1}{2}(v-1), 0, \ldots, 0]$. So the 1-rotational STS(v) is also a reverse STS(v) and so $v \equiv 1, 3, 9$ or $19 \pmod{24}$. Combining these conditions on v, we get $v \equiv 3$ or $9 \pmod{24}$.

Theorem 4.2. If $v \equiv 3$ or $9 \pmod{24}$ then there exists a 1-rotational STS(v).

Proof: Let $v \equiv 3$ or $9 \pmod{24}$ and let $X = Z_{v-1} \cup \{\infty\}$. Define a set β of blocks on X as follows: $\beta = \beta_1 \cup \beta_2$ where

$$\beta_1 = \{(\infty, i, i + \tfrac{1}{2}(v-1)) \mid i = 0, 1, \ldots, \tfrac{1}{2}(v-3)\}$$

$$\beta_2 = \{(i, i+r, i + b_r+n) \mid i = 1, 2, \ldots, v-2;\ r = 1, 2, \ldots, n\}$$

where $\{(a_r, b_r) \mid r = 1, 2, \ldots, n\}$ is any (A,n)-system with $n = (v-3)/6$. Since $v \equiv 3$ or $9 \pmod{24}$, $n \equiv 0$ or $1 \pmod 4$ such an (A,n)-system exists. These blocks satisfy the conditions for a STS(v) that admits the automorphism $\alpha = (\infty)(0, 1, \ldots, v-2)$.

So a 1-rotational STS(v) exists if and only if $v \equiv 3$ or $9 \pmod{24}$.

Example 4.1. Suppose $n = 13$ and $v = 81$. Then theorem 4.2 produces a 1-rotational STS(81) based on the (A,13)-system of example 2.1 with blocks in $\beta = \beta_1 \cup \beta_2$ where β_1 and β_2 are as follows:

$\beta_1 = \{(\infty, 0, 40)$ and its images under $\alpha = (\infty)(0, 1, \ldots, 79)\}$; $\beta_2 = \{(0, 1, 18), (0, 2, 34), (0, 3, 22), (0, 4, 35), (0, 5, 26), (0, 6,$

36), (0, 7, 23) (0, 8, 37), (0, 9, 24), (0, 10, 38), (0, 11, 25), (0, 12, 39), (0, 13, 33) and their images under α}. All blocks containing ∞ or a pair with difference 40 are contained in β_1. The blocks in β_2 contain any pair whose difference is in the set $\{r, a_r + 13, b_r + 13 \mid r = 1, 2, \ldots, 13\} = \{1, 2, \ldots, 39\}$. So all desired differences are covered. Similarly, in general, β_1 has all blocks containing ∞ and any pair with difference $(v-1)/2$ is in a block of β_1. Any pair with difference in the set $\{r, a_r + n, b_r + n \mid r = 1, 2, \ldots, n\} = \{1, 2, \ldots, (v-3)/2\}$ is in a triple of β_2.

Theorem 4.3. If there exists a 2-rotational STS(v) then $v \not\equiv 13$ or $21 \pmod{24}$.

Proof: Let $X = \{\infty, 0_1, 1_1, \ldots, [\frac{1}{2}(v-1)]_1, 0_2, 1_2, \ldots, [\frac{1}{2}(v-1)]_2\}$ and let $\alpha = (\infty)(0_1, 1_1, \ldots, [\frac{1}{2}(v-1)]_1)(0_2, 1_2, \ldots, [\frac{1}{2}(v-1)]_2)$ be an automorphism of a 2-rotational STS(v) with point set X. If $\frac{1}{2}(v-1) \equiv 0 \pmod 2$ then $\alpha^{(v-1)/4}$ is a permutation of type $[1, \frac{1}{2}(v-1), 0, \ldots, 0]$ and the STS(v) is a reverse STS(v). But a reverse STS(v) does not exist for $v \equiv 13$ or $21 \pmod{24}$.

Theorem 4.4. If there exists a cyclic STS(v) then there exists a 2-rotational STS(2v+1).

Proof: Let $V = Z_v$ and $\gamma = (0, 1, \ldots, v-1)$ be the point-set and automorphism, respectively, of a cyclic STS(v) with B as its set of blocks. Now let $W = \{\infty, 0_1, 1_2, \ldots, (v-1)_1, 0_2, 1_2, \ldots, (v-1)_2\}$. Define $C = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{(\infty, i_1, i_2) \mid i = 0, 1, \ldots, v-1\}$$

$$C_2 = \{(i_1, (i-j)_2, (i+j)_2) \mid i = 0, 1, \ldots, v-1;\ j = 1, 2, \ldots, (v-1)/2\}$$

$$C_3 = \{(i_1, j_1, k_1) \mid (i,j,k) \in B\}.$$

Now, the set W is a point-set and the set C is a set of blocks for a 2-rotational STS(2v+1) with automorphism $\alpha = (\infty)(0_1, 1_1, \ldots, (v-1)_1)\ (0_2, 1_2, \ldots, (v-1)_2)$.

<u>Theorem 4.5</u>. Any 1-rotational STS(v) is also 2-rotational.

<u>Proof</u>: A 1-rotational STS(v) exists if and only if $v \equiv 3$ or $9 \pmod{24}$, and so $\frac{1}{2}(v-1) \equiv 0 \pmod 2$. If α is a permutation of type $[1, 0, \ldots, 0, 1, 0]$ of a 1-rotational STS(v) then α^2 is a permutation of type $[1, 0, \ldots, 0, 2, 0, \ldots, 0]$ and so the STS(v) is also 2-rotational.

<u>Theorem 4.6</u>. Let n be a natural number, and let

$$S(n) = \{1, 2, \ldots, 2n-1, 2n+1, 2n+2, \ldots, 4n-1\}$$

$$T(n) = \begin{cases} \{2, 3, 4, \ldots, 2n\} & \text{if } n \text{ is odd} \\ \{1, 3, 4, \ldots, 2n\} & \text{if } n \text{ is even} \end{cases}$$

A set of $2n-1$ ordered pairs $\{(c_r, d_r) \mid r \in T(n)\}$ such that

$d_r - c_r = r$ for all $r \in T(n)$ and $\bigcup_{r \in T(n)} \{c_r, d_r\} = S(n)$ is called an _(F,n)-system_. An (F,n)-system exists if and only if $n \neq 2$.

Proof: $T(2) = \{1, 3, 4\}$, but the set $S(2) = \{1, 2, 3, 5, 6, 7\}$ cannot be partitioned into three pairs having differences 1, 3, and 4, so an (F,2)-system does not exist.

For $n = 1, 3, 4$, and 5, an (F,n)-system is:

$n = 1$: (1,3),

$n = 3$: (1,3), (8,11), (5,9), (2,7), (4,10),

$n = 4$: (13,14), (3,6), (11,15), (2,7), (4,10), (5,12), (1,9),

$n = 5$: (6,8), (16,19), (14,18), (12,17), (1,7) (2,9), (3,11), (4,13), (5,15).

Now, suppose $n \geq 6$.

Case 1. n-even, say $n = 2s$, $n \geq 6$. Consider the pairs:

$(r+1,\ 2n-r)$ $\quad r = 1, \ldots, n-2$

$(2n+1+r,\ 4n-1-r)$ $\quad r = 1, \ldots, s-2$

$(5s-1+r,\ 7s-1-r)$ $\quad r = 1, \ldots, s-2$

$(1,\ 2n+1)$

$(n,\ 3n-2)$

$(n+1,\ 3n)$

$(3n-1,\ 4n-1)$

$(7s-1,\ 7s)$

These pairs form an (F,n)-system.

Case 2. $n \equiv 3 \pmod 4$, say $n = 4s+3$, $n \geq 7$.

Consider the pairs:

$(r+1,\ 2n-r)$ $\quad r = 1, 2, \ldots, n-2$

$(2n+2r,\ 4n-2-2r)$ $\quad r = 1, 2, \ldots, 2s$

$(2n+1+2r,\ 4n+1-2r)$ $\quad r = 1, 2, \ldots, s$

$(2n+2s+1+2r,\ 3n+2s-2r)$ $\quad r = 1, 2, \ldots, s-1$ (omit if $s = 1$)

$(1,\ 2n+1)$

$(n,\ 3n-2)$

$(n+1, 3n)$

$(3n-1,\ 4n-2)$

$(3n+2s,\ 3n+2s+2)$

These pairs form an (F,n)-system.

Case 3. $n \equiv 1 \pmod 4$, say $n = 4s+1$, $n \geq 9$.

Consider the pairs:

$(r,\ 2n-1-r)$ $\quad r = 1, 2, \ldots, n-2$

$(n-1,\ 3n-2)$

$(n,\ 3n)$

$(2n-1,\ 4n-3)$

along with

a) if $s = 2$, the pairs

(19,29), (20,32), (21,35), (22,24), (23,31), (26,30), (28,34);

b) if $s \geq 3$, the pairs

$(2n-1+2r,\ 4n-3-2r)$ $\quad r = 1, 2, \ldots, s$

$(2n+2r,\ 4n-2r)$ $\quad r = 1, 2, \ldots, s$

$(2n+2s+3+2r,\ 3n+2s-2-2r)$ $r = 1, 2, \ldots, s-3$

(omit if $s = 2$ or 3)

$(2n+2s+2+4r,\ 3n+2s+3-4r)$ $r = 1, 2, \ldots, \frac{1}{2}(s-1)$

$(2n+2s+4r,\ 3n+2s-3-4r)$ $r = 1, 2, \ldots, \frac{1}{2}(s-2)$

(omit if $s = 2$ or 3)

$(3n+2,\ 4n-1)$

$(2n+2s+1,\ 2n+2s+3)$

$(3n-3,\ 3n+1)$ if s is odd

$(3n-1,\ 3n+3)$ if s is even.

These pairs form an (F,n)-system.

Example 4.2. Suppose $n = 8$. Then theorem 4.6 yields the following sets: $S(8) = \{1, 2, \ldots, 15, 17, 18, \ldots, 31\}$; $T(8) = \{1, 3, 4, \ldots, 16\}$; and the following pairs: (27,28), (7,10), (21,25), (6,11), (20,26), (5,12), (23,31), (4,13), (19,29), (3,14), (18,30), (2,15), (8,22), (9,24), and (1,17).

Theorem 4.7. If $v \equiv 1 \pmod{24}$, there exists a 2-rotational STS(v).

Proof: Let $v = 24t + 1$, let $X = \{\infty, 0_1, 1_1, \ldots, (12t-1)_1, 0_2, 1_2, \ldots, (12t-1)_2\}$, and let $\alpha = (\infty)(0_1, 1_1, \ldots, (12t-1)_1)(0_2, 1_2, \ldots, (12t-1)_2)$. Define a set of blocks $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$, where B_i is defined as follows:

$B_1 = \{(\infty, 0_1, (6t)_1), (\infty, 0_2, (6t)_2\}$

$B_2 = \{(0_1, (4t)_1, (8t)_1)\}$

$B_3 = \{(0_1, r_1, (b_r-1)_2) \mid r = 1, 2, \ldots, 6t-1; r \neq 4t\}$

where $\{(a_r,b_r) \mid r = 1, 2, \ldots, 6t-1\}$ is an $(A,6t-1)$-system or a $(B,6t-1)$-system depending on whether t is odd or even,

$B_4 = \{(0_1, (a_{4t}-1), (b_{4t}-1)_2\}$

$$B_5 = \begin{cases} \{(0_2, 1_2, 2_1)\} & \text{if } t \text{ is odd} \\ \{(0_2, 2_2, 3_1)\} & \text{if } t \text{ is even} \end{cases}$$

$$B_6 = \begin{cases} \{(0_2, 1_2, 10_2), (0_2, 5_2, 11_2), (0_2, 3_2, 7_2)\} & \text{if } t = 2 \\ \{(0_2, (c_r + 2t)_2, (d_r + 2t)_2) \mid r \in T(t)\} & \text{if } t \neq 2 \end{cases}$$

where $\{(c_r,d_r) \mid r \in T(t)\}$ is an (F,t)-system.

The set B of blocks is a set of base blocks for a 2-rotational STS(v) under the automorphism α.

<u>Example 4.3</u>. Suppose $v = 97$ and $t = 4$. Then according to theorem 4.7, a 2-rotational STS(97) can be constructed on the set $\{\infty, 0_1, 1_1, \ldots, 47, 0_2, 1_2, \ldots, 47_2\}$. Define x as a <u>pure difference of type i (mod N)</u> if it is the difference (mod N) associated with the elements (y,z) of a subscripted pair of type (y_i, z_i). In this example, the difference associated with the pair (y_2,z_1) will be $(y-z) \bmod (\frac{1}{2}(v-1))$ and will be called a <u>mixed difference</u>. Theorem 4.7 yields the following collections of blocks: $B_1 = \{(\infty,0_1,24_1), (\infty,0_2,24_2)\}$, $B_2 = \{(0_1,16_1,32_1)\}$, $B_3 =$

$\{(0_1,1_1,6_2), (0_1,2_1,35_2), (0_1,3_1,13_2), (0_1,4_1,36_2), (0_1,5_1,14_2),$ $(0_1,6_1,37_2), (0_1,7_1,15_2), (0_1,8_1,38_2), (0_1,9_1,16_2), (0_1,10_1,39_2),$ $(0_1,11_1,22_2), (0_1,12_1,40_2), (0_1,13_1,17_2), (0_1,14_1,41_2), (0_1, 15_1, 18_2),$ $(0_1, 17_1, 19_2), (0_1, 18_1, 43_2), (0_1,19_1,20_2), (0_1,20_1,44_2),$ $(0_1,21_1,21_2), (0_1,22_1,34_2), (0_1,23_1,46_2)\}$, $B_4 = \{(0_1,26_2,42_2)\}$, $B_5 = \{(0_2,2_2,3_1)\}$, and $B_6 = \{(0_2,21_2,22_2), (0_2,11_2,14_2), (0_2,19_2,23_2), (0_2,10_2,15_2), (0_2,12_2,18_2), (0_2,13_2,20_2), (0_2,9_2,17_2)\}$. All blocks containing ∞ are generated by the blocks of B_1. Also, the blocks of B_1 cover the pure difference of 24. The block of B_2 covers the pure difference of type 1 equal to 16. The blocks of B_3 cover the pure differences of type 1 in the set $\{1, 2, ..., 15, 17, 18, ..., 23\}$ and the mixed differences in the set $\{a_r-1, b_r-1 \mid r = 1, 2, ..., 15, 17, 18, ..., 23\} = \{0, ..., 44, 46\} - \{26,42\}$ where the a_r's and b_r's are from the $(B,23)$-system from example 2.3. The block in B_4 covers the mixed differences 26 and 42 and the pure difference of type 2 equal to 16. The block in B_5 covers the pure difference of type 2 equal to 2 and the mixed differences 45 and 47. The blocks in B_6 cover the pure differences of type 2 in the sets $T(4)$ and $\{x + 8 \mid x \in S(4)\}$. So all desired differences are covered. In general, all blocks containing ∞ are generated by the blocks of B_1 and these blocks cover the pure differences of $6t$. The block of B_2 covers the pure difference of type 1 equal to $4t$. The blocks of B_3 cover the pure differences of type 1 in the set $\{1, 2, \ldots, 4t-1, 4t+1, 4t+2, \ldots, 6t-1\}$ and the mixed differences in the set $\{a_r-1, b_r-1 \mid r = 1, 2, \ldots, 4t-1, 4t+1, 4t+2, \ldots, 6t-1\} = \{0, 1, \ldots, 12t - 4\} \cup A - \{a_{4t}-1, b_{4t}-1\}$ where $A = \{12t - 3\}$ if t is odd or $A = \{12t - 2\}$ if t is even. The block in B_4 covers the mixed differences $a_{4t}-1$ and $b_{4t}-1$ and the pure difference of type 2 equal to 4t. The block in B_5 covers the pure difference of type 2 equal to 1 if t is odd or equal to 2 if t is even. This block also covers the mixed differences equal to 1 and 2 if t is odd or equal to 1 or 3 if t is even. The blocks in B_6 cover the pure differences of type 2 in the sets $T(t)$ and $\{x + 2t \mid x \in S(t)\}$, where $T(t)$ and $S(t)$ are as described in theorem 4.6. So all desired differences are covered.

Theorem 4.8. A 2-rotational STS(v) exists if and only if $v \equiv 1, 3, 7, 9, 15$ or $19 \pmod{24}$.

Proof: The condition on v is shown to be necessary in theorem 4.3. The sufficiency for $v \equiv 3$ or $9 \pmod{24}$ follows from theorem 4.2 and theorem 4.5. Since a cyclic STS(v) exists for all $v \equiv 1$ or $3 \pmod 6$ except $v = 9$, by theorem 4.4, there exists a 2-rotational STS(v) for all $v \equiv 3$ or $7 \pmod{12}$, except possibly for $v = 19$. Theorem 4.7 demonstrates the existence for $v \equiv 1 \pmod{24}$.

Finally, the blocks:

$(\infty, 0_1, 0_2)$, $(0_1, 3_1, 6_1)$, $(0_2, 1_2, 3_2)$,

$(5_1, 0_2, 4_2)$, $(3_1, 4_1, 0_2)$, $(6_1, 8_1, 0_2)$,

$(2_1, 7_1, 0_2)$

are base blocks for a 2-rotational STS(19) under the automorphism $\alpha = (\infty)(0_1, 1_1, \ldots, 9_1)(0_2, 1_2, \ldots, 9_2)$.

Theorem 4.9. A 6-rotational STS(v) exists if and only if $v \equiv 1, 7,$ or 19 (mod 24).

Proof: If there exists a 6-rotational STS(v), then it admits an automorphism α of type $[1, 0, \ldots, 0, 6, 0, \ldots, 0]$, that is, an automorphism consisting of a fixed point and 6 cycles of length $(v-1)/6$ each. So $v \equiv 1 \pmod 6$ and $v \equiv 1, 7, 13,$ or 19 (mod 24). If $v \equiv 13 \pmod{24}$ then $v-1 \equiv 0 \pmod{12}$ and the STS(v) admits $\alpha^{(v-1)/12}$ which is an automorphism of type $[1, (v-1)/2, 0, \ldots, 0]$ and so the STS(v) is also reverse. But there are no reverse STS(v) for $v \equiv 13 \pmod{24}$.

However, if $v \equiv 1, 7$ or 19 (mod 24) then $v-1 \equiv 0 \pmod 6$ and there exists a 2-rotational STS(v) admitting an automorphism β of type $[1, 0, \ldots, 0, 2, 0, \ldots, 0]$. Now β^3 is an automorphism of this STS(v) and of type $[1, 0, \ldots, 0, 6, 0, \ldots, 0]$. So the STS(v) is also 6-rotational.

Cho [2] presented the results for 3-rotational and 4-rotational systems.

Theorem 4.10. The set $\{1, 3, 4, \ldots, 2n+1\}$ can be partitioned into pairs (a_r, b_r), $b_r = a_r + r$, $r = 1, 2, \ldots, n$ if and only if $n \equiv 2$ or $3 \pmod 4$. Such a partitioning is called a (-B,n)-system.

Proof: If there is such a partitioning, then $b_r - a_r = r$ for $r = 1, 2, \ldots, n$ and $\Sigma b_r - \Sigma a_r = \frac{1}{2}n(n+1)$. Since the a_r's and b_r's partition the set $\{1, 3, 4, \ldots, 2n+1\}$, $\Sigma a_r + \Sigma b_r = 2n^2 + 3n - 1$. So $\Sigma b_r = \dfrac{5n^2 + 7n-2}{4}$ and $n \equiv 2$ or $3 \pmod 4$ is a necessary condition.

So suppose $n \equiv 2 \pmod 4$, say $n = 4s + 2$, $s \geq 1$. Consider the pairs:

$(s + 3 + r, 3s + 4 - r)$ $r = 1, 2, \ldots, s-1$ (omit if $s = 1$)

$(2 + r, 4s + 3 - r)$ $r = 1, 2, \ldots, s-1$ (omit if $s = 1$)

$(4s + 4 + r, 8s + 6 - r)$ $r = 1, 2, \ldots, 2s$

$(s + 2, s + 3)$

$(1, 4s + 3)$

$(2s + 3, 4s + 4)$

$(2s + 4, 6s + 5)$.

For $n = 2$, take the pairs $(1,3)$ and $(4,5)$.

Now suppose $n \equiv 3 \pmod 4$, say $n = 4s - 1$, $s \geq 2$.

Consider the pairs:

$(5s + r, 7s - 1 - r)$ $r = 1, 2, \ldots, s-2$ (omit if $s = 2$)

$(2 + r, 4s - r)$ $r = 1, 2, \ldots, 2s-2$

$(4s + 1 + r, 8s - r)$ $r = 1, \ldots, s-1$

$(7s - 1, 7s)$

$(1, 4s)$

$(2s + 1, 6s - 1)$

$(4s + 1, 6s)$.

For $n = 3$ take the pairs (6,7), (3,5), and (1,4). These pairs satisfy the conditions of a $(-B,n)$-system.

Example 4.4. Theorem 4.10 produces a $(-B,18)$-system with the following pairs: (6,7), (28,30), (10,13), (27,31), (9,14), (26,32), (8,15), (25,33), (11, 20), (24,34), (5, 16), (23, 35), (4, 17), (22,36), (3,18), (21,37), (12,29), and (1,19).

Theorem 4.11. If a 3-rotational STS(v) exists, then $v \equiv 1$ or 19 (mod 24).

Proof: Let α be an automorphism of a 3-rotational STS(v), where α is of type [1, 0, ..., 0, 3, 0, ..., 0]. Now, $(v-1)/3$ is an integer, so $v \equiv 1 \pmod 3$. Combining this with the condition that $v \equiv 1$ or 3 (mod 6), it follows that $v \equiv 1$, 7, 13 or 19 (mod 24). If $v \equiv 7$ or 13 (mod 24), then $v - 1 \equiv 0 \pmod 6$. So $\alpha^{(v-1)/6}$ is a permutation of type $[1, \frac{1}{2}(v-1), 0, \ldots, 0]$. So the 3-rotational STS(v) is also a reverse STS(v). But a reverse STS(v) does not exist for $v \equiv 7$ or 13 (mod 24). So $v \equiv 1$ or 19 (mod 24).

Theorem 4.12. The set $\{1, 2, \ldots, \frac{1}{2}(n+1)-1, \frac{1}{2}(n+1)+1, \ldots, 2n+1\}$ can be partitioned into pairs (a_r, b_r) for $r = 1, 2, \ldots, n$ if and only if $n \equiv 1$ or $3 \pmod 4$. Such a partitioning is called an (E,n)-system.

Proof: Since $\frac{1}{2}(n+1)$ is an integer, $n \equiv 1$ or $3 \pmod 4$.

So suppose $n \equiv 1 \pmod 4$, say $n = 4s + 1$.

Consider the pairs:

$(4s + 1 + r, 8s + 4 - r)$ $r = 1, 2, \ldots, 2s+1$

$(r, 4s + r - r)$ $r = 1, 2, \ldots, 2s$.

Now suppose $n \equiv 3 \pmod 4$, say $n = 4s - 1$.

Consider the pairs;

$(4s - 1 + r, 8s - r)$ $r = 1, 2, \ldots, 2s$

$(r, 4s - r)$ $r = 1, 2, \ldots, 2s - 1$.

These pairs satisfy the conditions of an (E,n)-system.

Example 4.5. Theorem 4.12 produces an (E,15)-system with the following pairs: (23,24), (7,9), (22,25), (6,10), (21,26), (5,11), (20,27), (4,12), (19,28), (3,13), (18,29), (2,14), (17,30), (1,15) and (16,31).

Theorem 4.13. If $v \equiv 1 \pmod{24}$ then there exists a 3-rotational STS(v).

<u>Proof</u>: Let $v = 24s + 1$, $s \geq 1$, let $X = \{\infty, 0_1, 1_1, \ldots, (8s - 1)_1, 0_2, 1_2, \ldots, (8s - 1)_2, 0_3, 1_3, \ldots, (8s - 1)_3\}$ and let $\alpha = (\infty)(0_1, 1_1, \ldots, (8s - 1)_1)\ (0_2, 1_2, \ldots, (8s - 1)_2)\ (0_3, 1_3, \ldots, (8s - 1)_3)$ be a permutation of X. Consider the set of blocks $B = B_1 \cup B_2 \cup B_3$ B_4 where:

$B_1 = \{(\infty, 0_i, (4s)_i) \mid i = 1, 2, 3\}$

$B_2 = \{(0_1, 0_2, 0_3), (0_1, (2s)_2, (6s)_3)\}$

$B_3 = \{(0_1, r_1, (b_r)_2), (0_3, r_3, (b_r)_1) \mid r = 1, 2, \ldots, 4s-1\}$

where (a_r, b_r), $r = 1, 2, \ldots, 4s - 1$ is an $(E, 4s - 1)$-system.

$B_4 = \{(0_2, r_2, (b_r)_3) \mid r = 1, 2, \ldots, 4s - 1\}$ where (a_r, b_r), $r = 1, 2, \ldots, 4s-1$ is a $(C, 4s - 1)$-system.

These blocks are the base blocks for a 3-rotational STS(v) under α.

<u>Example 4.6</u>. Suppose $v = 97$ and $s = 4$. Then according to theorem 4.13, a 3-rotational STS(97) can be constructed on the set $\{\infty, 0_1, 1_2, \ldots, 31_1, 0_2, 1_2, \ldots, 31_2, 0_3, 1_3, \ldots, 31_3\}$. Theorem 4.13 yields the following collections of base blocks: $B_1 = \{(\infty, 0_1, 16_1), (\infty, 0_2, 16_2), (\infty, 0_3, 16_3)\}$; $B_2 = \{(0_1, 0_2, 0_3), (0_1, 8_2, 24_3)\}$; $B_3 = \{(0_1, 1_1, 24_2), (0_1, 2_1, 9_2), (0_1, 3_1, 25_2), (0_1, 4_1, 10_2), (0_1, 5_1, 26_2), (0_1, 6_1, 11_2), (0_1, 7_1, 27_2), (0_1, 8_1, 12_2), (0_1, 9_1, 28_2), (0_1, 10_1, 13_2), (0_1, 11_1, 29_2), (0_1, 12_1, 14_2), (0_1, 13_1, 30_2), (0_1,$

14_1, 15_2), (0_1, 15_1, 31_2), (0_3, 1_3, 24_1), (0_3, 2_3, 9_1), (0_3, 3_3, 25_1), (0_3, 4_3, 10_1), (0_3, 5_3, 26_1), (0_3, 6_3, 11_1), (0_3, 7_3, 27_1), (0_3, 8_3, 12_1), (0_3, 9_3, 28_1), (0_3, 10_3, 13_1), (0_3, 11_3, 29_1), (0_3, 12_3, 14_1), (0_3, 13_3, 30_1), (0_3, 14_3, 15_1), (0_3, 15_3, 31_1)}; B_4 = {(0_2, 1_2, 28_3), (0_2, 2_2, 9_3), (0_2, 3_2, 25_3), (0_2, 4_2, 10_3), (0_2, 5_2, 26_3), (0_2, 6_2, 11_3), (0_2, 7_2, 24_3), (0_2, 8_2, 12_3), (0_2, 9_2, 29_3) (0_2, 10_2, 13_3), (0_2, 11_2, 30_3), (0_2, 12_2, 14_3), (0_2, 13_2, 31_3), (0_2, 14_2, 15_3), (0_2, 15_2, 23_3)}. All blocks containing ∞ are generated by the blocks of B_1. Also the blocks of B_1 cover all pure differences equal to 16. The blocks of B_2 cover all mixed differences of 0 and contains a pair (x_2,y_1) where $x - y \equiv 8 \pmod{32}$, a pair (x_3,y_2) where $x - y \equiv 16 \pmod{32}$, and a pair (x_1,y_3) where $x - y \equiv 8 \pmod{32}$. The blocks of B_3 cover all pure differences of types 1 and 3 in the set {1, 2, ..., 15} and contains pairs (x_2,y_1) and (x_1,y_3) where $(x-y) \pmod{32}$ is the set $\{a_r, b_r \mid r = 1, 2, \ldots, 15\}$ = {1, 2, ..., 7, 9, 10, ..., 31} where the a_r and b_r are from the (E,15)-system of example 4.5. The blocks of B_4 cover all pure differences of type 2 in the set {1, 2, ..., 15} and contains pairs (x_3, y_2) where $(x - y) \pmod{32}$ is in the set $\{a_r, b_r \mid r = 1, 2, \ldots 15\}$ = {1, 2, ..., 15, 17, 18, ..., 31} where the a_r and b_r are from the (C,15)-system of example 2.5. So all desired differences are covered and the collections of blocks actually form a set of base blocks. In general, all blocks containing ∞ are generated by the blocks of B_1. Also the blocks of B_1 cover all pure differences of 4s. The blocks of B_2 cover all mixed differences equal to 0, and contains a pair (x_2, y_1) where $(x-y) \equiv 2s \pmod{8s}$, a pair (x_3, y_2) where $(x - y) \equiv 4s \pmod{8s}$, and a pair (x_1, y_3) where $(x - y) \equiv 2s \pmod{8s}$. The blocks of B_3 cover all pure differences of type 1 in the set $\{1, 2, \ldots, 4s - 1\}$ and all pairs (x_2, y_1) where $(x - y) \pmod{8s}$ is in the set $\{a_r, b_r \mid r = 1, 2, \ldots, 4s - 1\} = \{1, 2, \ldots, 2s - 1, 2s + 1, 2s + 2, \ldots, 8s - 1\}$ where the a_r and b_r are from an (E, 4s - 1)-system . The blocks of B_3 also contain all pure differences of type 3 in the set $\{1, 2, \ldots, 4s - 1\}$ and contains pairs (x_1, y_3) where $(x - y) \pmod{8s}$ is in the set $\{a_r, b_r \mid r = 1, 2, \ldots, 4s - 1\} = \{1, 2, \ldots, 2s - 1, 2s + 1, 2s + 2, \ldots, 8s - 1\}$ where the a_r and b_r are from an (E, 4s - 1) system from theorem 4.12. The blocks of B_4 cover all pure differences of type 2 in the set $\{1, 2, \ldots, 4s - 1\}$ and contains pairs (x_3, y_2) where $(x - y) \pmod{8s}$ is in the set $\{a_r, b_r \mid r = 1, 2, \ldots, 4s - 1\} = \{1, 2, \ldots, 4s - 1, 4s + 1, 4s + 2, \ldots, 8s - 1\}$ where the a_r and b_r are from a (C, 4s - 1)-system.

Theorem 4.14. If $v \equiv 19 \pmod{24}$ then there exists a 3-rotational STS(v).

Proof. Let $v = 24s + 19$, $s \geq 0$, let $X = \{\infty, 0_1, 1_1, \ldots, (8s+5)_1, 0_2, 1_2, \ldots, (8s+5)_2, 0_3, 1_3, \ldots, (8s+5)_3\}$ and let $\alpha = (\infty)(0_1, 1_1, \ldots, (8s+5)_1)(0_2, 1_2, \ldots, (8s+5)_2)(0_3, 1_3, \ldots, (8s+5)_3)$ be a permutation of X. Consider the set of blocks $B = B_1 \cup B_2 \cup B_3 \cup B_4$ where

$B_1 = \{(\infty, 0_i, (4s+3)_i) \mid i = 1, 2, 3\}$

$B_2 = \{(0_1, (4s+3)_2, (4s+3)_3), (0_1, (8s+5)_2, 1_3)\}$

$B_3 = \{(0_1, r_1, (b_r)_2), (0_3, r_3, (b_r)_1) \mid r = 1, 2, \ldots, 4s+2\}$

where (a_r, b_r), $r = 1, 2, \ldots, 4s+2$ is a $(D, 4s+2)$-system.

$B_4 = \{(0_2, r_2, (b_r)_3) \mid r = 1, 2, \ldots, 4s+2\}$ where (a_r, b_r), $r = 1, 2, \ldots, 4s+2$ is a $(-B, 4s+2)$-system.

These blocks are the base blocks for a 3-rotational STS(v) under α. So a 3-rotational STS(v) exists if and only if $v \equiv 1$ or $19 \pmod{24}$.

Example 4.7. Suppose $v = 115$ and $s = 4$. Then according to theorem 4.14, a 3-rotational STS(115) can be constructed on the set $\{\infty, 0_1, 1_1, \ldots, 37_1, 0_2, 1_2, \ldots, 37_2, 0_3, 1_3, \ldots, 37_3\}$. Theorem 4.14 yields the following collections of base blocks: $B_1 = \{(\infty, 0_1, 19_1), (\infty, 0_2, 19_2), (\infty, 0_3, 19_3)\}$; $B_2 = \{(0_1, 19_2, 19_3), (0_1, 37_2, 1_3)\}$; $B_3 = \{(0_1, 1_1, 32_2), (0_1, 2_1, 38_2), (0_1, 3_1, 11_2), (0_1, 4_1, 29_2), (0_1, 5_1, 12_2),$

$(0_1, 6_1, 30_2)$, $(0_1, 7_1, 13_2)$, $(0_1, 8_1, 28_2)$, $(0_1, 9_1, 14_2)$, $(0_1, 10_1, 33_2)$, $(0_1, 11_1, 15_2)$, $(0_1, 12_1, 34_2)$, $(0_1, 13_1, 16_2)$, $(0_1, 14_1, 35_2)$, $(0_1, 15_1, 17_2)$, $(0_1, 16_1, 26_2)$, $(0_1, 17_1, 18_2)$, $(0_1, 18_1, 27_2)$, $(0_3, 1_3, 32_1)$, $(0_3, 2_3, 38_1)$, $(0_3, 3_3, 11_1)$, $(0_3, 4_3, 29_1)$, $(0_3, 5_3, 12_1)$, $(0_3, 6_3, 30_1)$, $(0_3, 7_3, 13_1)$, $(0_3, 8_3, 28_1)$, $(0_3, 9_3, 14_1)$, $(0_3, 10_3, 33_1)$, $(0_3, 11_3, 15_1)$, $(0_3, 12_3, 34_1)$, $(0_3, 13_3, 16_1)$, $(0_3, 14_3, 35_1)$, $(0_3, 15_3, 17_1)$, $(0_3, 16_3, 26_1)$, $(0_3, 17_3, 18_1)$, $(0_3, 18_3, 27_1)\}$;

$B_4 = \{(0_2, 1_2, 7_3)$, $(0_2, 2_2, 30_3)$, $(0_2, 3_2, 13_3)$, $(0_2, 4_2, 31_3)$, $(0_2, 5_2, 14_3)$, $(0_2, 6_2, 32_3)$, $(0_2, 7_2, 15_3)$, $(0_2, 8_2, 33_3)$, $(0_2, 9_2, 20_3)$, $(0_2, 10_2, 34_3)$, $(0_2, 11_2, 16_3)$, $(0_2, 12_2, 35_3)$, $(0_2, 13_2, 17_3)$, $(0_2, 14_2, 5_3)$, $(0_2, 15_2, 18_3)$, $(0_2, 16_2, 37_3)$, $(0_2, 17_2, 29_3)$, $(0_2, 18_2, 19_3)\}$. All blocks containing ∞ are generated by the blocks of B_1. Also, the blocks of B_1 cover all pure differences equal to 19. The blocks of B_2 contain pairs (x_2, y_1) with $x - y \equiv 19 \pmod{38}$ and $x - y \equiv 37 \pmod{38}$, pairs (x_1, y_3) with $x - y \equiv 37 \pmod{38}$ and $x - y \equiv 19 \pmod{38}$, and pairs (x_3, y_2) with $x - y \equiv 0 \pmod{38}$ and $x - y \equiv 2 \pmod{38}$. The blocks of B_3 cover all pure differences of types 1 and 3 in the set $\{1, 2, \ldots, 18\}$ and contains pairs (x_2, y_1) and (x_1, y_3) where $(x - y) \pmod{38}$ is in the set

$\{a_r, b_r \mid r = 1, 2, \ldots, 18\} = \{1, 2, \ldots, 18, 20, 21, \ldots, 36, 38\}$

where the a_r and b_r are from the $(D,18)$-system of example 2.6. The blocks of B_4 cover all pure differences of type 2 in the set $\{1, 2, \ldots, 18\}$ and contains pairs (x_3, y_2) where $(x - y) \pmod{38}$

is in the set $\{a_r, b_r \mid r = 1, 2, \ldots, 18\} = \{1, 3, 4, \ldots, 37\}$ where the a_r and b_r are from the $(-B,18)$-system of example 4.4. In general, all blocks containing ∞ are generated by the blocks of B_1. Also the blocks of B_1 cover all pure differences equal to $4s + 3$. The blocks of B_2 contain pairs (x_2, y_1) with $x - y \equiv 4s + 3 \pmod{8s + 6}$ and $x - y \equiv 8s + 5 \pmod{8s + 6}$, pairs (x_1, y_3) with $x - y \equiv 8s + 5 \pmod{8s + 6}$ and $x - y \equiv 4s + 3 \pmod{8s + 6}$, and pairs (x_3, y_2) with $x - y \equiv 0 \pmod{8s + 6}$, and $x - y \equiv 2 \pmod{8s + 6}$. The blocks of B_3 cover all pure differences of types 1 and 3 in the set $\{1, 2, \ldots, 4s + 2\}$ and contains pairs (x_2,y_1) and (x_1,y_3) where $(x - y) \pmod{8s + 6}$ is in the set $\{a_r,b_r \mid r = 1, 2, \ldots, 4s + 2\}$ $= \{1, 2, \ldots, 4s + 2, 4s + 4, 4s + 5, \ldots, 8s + 4, 8s + 6\}$ where the a_r and b_r are from a $(D, 4s + 2)$-system. The blocks of B_4 cover all pure differences of type 2 in the set $\{1, 2, \ldots, 4s + 2\}$ and contains pairs (x_3, y_2) where $(x - y) \pmod{8s + 6}$ is in the set $\{a_r, b_r \mid r = 1, 2, \ldots, 4s + 2\} = \{1, 3, 4, \ldots, 8s + 5\}$ where the a_r and b_r are from a $(-B, 4s + 2)$ - system.

<u>Theorem 4.15</u>. If a 4-rotational $STS(v)$ exists, then $v \equiv 1$ or $9 \pmod{12}$.

<u>Proof</u>. Since $v \equiv 1$ or $3 \pmod 6$ and $v - 1 \equiv 0 \pmod 4$, that is $v \equiv 1 \pmod 4$, it follows that $v \equiv 1$ or $9 \pmod{12}$.

Theorem 4.16. If $v \equiv 1$ or $9 \pmod{24}$, then there exists a 4-rotational STS(v).

Proof. If $v \equiv 1$ or $19 \pmod{24}$, then there exists a 2-rotational STS(v) with an automorphism α of type $[1, 0, \ldots, 0, 2, 0, \ldots, 0]$ and α^2 is an automorphism of type $[1, 0, \ldots, 0, 4, 0, \ldots, 0]$. So the STS(v) is also 4-rotational.

Theorem 4.17. If $v \equiv 13 \pmod{24}$, then there exists a 4-rotational STS(v).

Proof. Let $v = 24s + 13$, $s \geq 2$, let $X = \{\infty, 0_1, 1_1, \ldots, (6s+2)_1, 0_2, 1_2, \ldots, (6s+2)_2, 0_3, 1_3, \ldots, (6s+2)_3, 0_4, 1_4, \ldots, (6s+2)_4\}$ and let $\alpha = (\infty)(0_1, 1_1, \ldots, (6s+2)_1)(0_2, 1_2, \ldots, (6s+2)_2)(0_3, 1_3, \ldots (6s+2)_3)(0_4, 1_4, \ldots, (6s+2)_4)$ be a permutation of X. Consider the set of blocks $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6 \cup B_7$ where B_i is as defined below.

Case 1: $s \equiv 0$ or $1 \pmod 4$. Let (a_r, b_r), $r = 1, 2, \ldots, 3s+1$ be an $(A, 3s+1)$-system.

$B_1 = \{(\infty, 0_1, (2s+1-b_{2s+1})_3), (\infty, 0_2, (-b_{2s+1})_4)\}$

B_2 = the set of all base blocks of a cyclic STS(6s + 3) with point set $\{0_3, 1_3, \ldots, (6s+3)_3\}$

$B_3 = \{(0_4, (2s+1)_4, (4s+2)_4\}$

$B_4 = \{(0_4, r_4, (b_r)_2) \mid r = 1, 2, \ldots, 2s, 2s+2, 2s+3, \ldots, 3s+1\}$

$B_5 = \{(0_1, r_1, (b_r)_2), (0_2, r_2, b_r)_3) \mid r = 1, 2, \ldots, 3s + 1\}$

$B_6 = \{(0_1, 0_2, (2s+1 - b_{2s+1})_4), (0_2, 0_3, 0_4)\}$

$B_7 = \{((b_{2s+1} - 2s - 1 - r)_1, 0_3, r_4) \mid r = 1, 2, \ldots, 6s + 2\}$

Case 2. $s \equiv 2$ or $3 \pmod 4$. Let (a_r, b_r), $r = 1, 2, \ldots, 3s + 1$ be a (B, 3s+1)-system.

$B_1 = \{(\infty, 0_1, (2s - 2 - b_{2s+1}0_3), (\infty, 0_2, (-b_{2s+1})_4)\}$

B_2 = the set of all base blocks of a cyclic STS(6s + 3) with point

set $\{0_3, 1_3, \ldots, (6s + 3)_3\}$

$B_3 = \{(0_4, (2s + 1)_4, (4s + 2)_4)\}$

$B_4 = \{(0_4, r_4, (b_r)_2) \mid r = 1, 2, \ldots, 2_s, 2s + 2, \ldots, 3s + 1\}$

$B_5 = \{(0_1, r_1, (b_r)_2), (0_2, r_2, (b_r)_3) \mid r = 1, 2, \ldots, 3s + 1\}$

$B_6 = \{(0_1, (6s + 2)_2, (2s - b_{2s+1})_4), (0_2, (6s + 2)_3, 1_4)\}$

$B_7 = \{((b_{2s+1} - 2s + 2 - r)_1, 0_3, (2 + r)_4) \mid r = 1, 2, \ldots, 6s + 2\}$

In either case, the blocks of B are the base blocks for a 4-rotational STS(v) under α.

Now, for v = 13, consider the blocks $(\infty, 0_1, 0_3)$, $(\infty, 0_2, 0_4)$, $(0_1, 0_2, 2_2)$, $(0_1, 1_1, 1_4)$, $(1_1, 0_3, 0_4)$, $(0_2, 0_3, 2_1)$, $(0_3, 2_2, 1_4)$, $(0_3, 1_2, 2_4)$, $(0_4, 1_4, 2_4)$ and $(2_3, 1_3, 0_3)$. These are the base blocks for a 4-rotational STS(13) under $\alpha = (\infty)(0_1, 1_1, 2_1)$ $(0_2, 1_2, 2_2)(0_4, 1_4, 2_4)$.

For v = 37, consider the blocks $(\infty, 0_1, 7_3)$, $(\infty, 0_2, 0_4)$, $(0_3, 3_3, 6_3)$, $(0_4, 3_4, 6_4)$, $(0_3, 1_3, 4_1)$, $(0_3, 2_3, 0_4)$, $(0_3, 4_3, 5_2)$, $(0_2, 0_3, 8_4)$, $(0_4, 1_4, 5_1)$, $(0_4, 2_4, 7_2)$, $(0_4, 4_4, 8_2)$, $(0_1, 0_2, 6_4)$,

$(0_1, 1_1, 2_2)$, $(0_1, 2_1, 7_2)$, $(0_1, 3_1, 6_2)$, $(0_1, 4_1, 8_2)$, $(0_2, 1_2, 2_3)$, $(0_2, 2_2, 7_3)$, $(0_2, 3_2, 6_3)$, $(0_2, 4_2, 7_4)$, $(0_1, 0_3, 1_4)$, $(0_1, 1_3, 3_4)$, $(0_1, 8_3, 2_4)$, $(0_1, 3_3, 7_4)$, $(0_1, 4_3, 0_4)$, and $(0_1, 2_3, 8_4)$. These are the base blocks for a 4-rotational STS(37) under $\alpha = (\infty)(0_1, 1_1, \ldots, 8_1)(0_2, 1_2, \ldots, 8_2)(0_3, 1_3, \ldots, 8_3)(0_4, 1_4, \ldots, 8_4)$.

<u>Example 4.8</u>. Suppose $v = 109$ and $s = 4$. Then according to Theorem 4.17, a 4-rotational STS(109) can be constructed on the set $\{\infty, 0_1, 1_1, \ldots, 26_1, 0_2, 1_2, \ldots, 26_2, 0_3, 1_3, \ldots, 26_3, 0_4, 1_4, \ldots, 26_4\}$. Theorem 4.17 yields the following collections of base blocks: $B_1 = \{(\infty, 0_1, 25_3), (\infty, 0_2, 16_4)\}$; $B_2 = \{(0_3, 1_3, 6_3), (0_3, 2_3, 13_3), (0_3, 3_3, 10_3), (0_3, 4_3, 12_3), (0_3, 9_3, 18_3)\}$; $B_3 = \{(0_4, 9_4, 18_4)\}$; $B_4 = \{(0_4, 1_4, 5_2), (0_4, 2_4, 21_2), (0_4, 3_4, 9_2), (0_4, 4_4, 22_2), (0_4, 5_4, 13_2), (0_4, 6_4, 23_2), (0_4, 7_4, 10_2), (0_4, 8_4, 24_2), (0_4, 10_4, 25_2), (0_4, 11_4, 12_2), (0_4, 12_4, 26_2), (0_4, 13_4, 20_2)\}$; $B_5 = \{(0_1, 1_1, 5_2), (0_1, 2_1, 21_2), (0_1, 3_1, 9_2), (0_1, 4_1, 22_2), (0_1, 5_1, 13_2), (0_1, 6_1, 23_2), (0_1, 7_1, 10_2), (0_1, 8_1, 24_2), (0_1, 9_1, 11_2), (0_1, 10_1, 25_2), (0_1, 11_1, 12_2), (0_1, 12_1, 26_2), (0_1, 13_1, 20_2), (0_2, 1_2, 5_3), (0_2, 2_2, 21_3), (0_2, 3_2, 9_3), (0_2, 3_2, 9_3), (0_2, 4_2, 22_3), (0_2, 5_2, 13_3), (0_2, 6_2, 23_3), (0_2, 7_2, 10_3), (0_2, 8_2, 24_3), (0_2, 9_2, 11_3), (0_2, 10_2, 25_3), (0_2, 11_2, 12_3), (0_2, 12_2, 26_3), (0_2, 13_2, 20_3)\}$; $B_6 = \{(0_1, 0_2, 25_4), (0_2, 0_3, 0_4)\}$; $B_7 = \{(1_1, 0_3, 1_4), (0_1, 0_3, 2_4), (27_1, 0_3, 3_4),$

$(26_1, 0_3, 4_4)$, $(25_1, 0_3, 5_4)$, $(24_1, 0_3, 6_4)$, $(23_1, 0_3, 7_4)$,

$(22_1, 0_3, 8_4)$, $(21_1, 0_3, 9_4)$, $(20_1, 0_3, 10_4)$, $(19_1, 0_3, 11_4)$,

$(18_1, 0_3, 12_4)$, $(17_1, 0_3, 13_4)$, $(16_1, 0_3, 14_4)$, $(15_1, 0_3, 15_4)$,

$(14_1, 0_3, 16_4)$, $(13_1, 0_3, 17_4)$, $(12_1, 0_3, 18_4)$, $(11_1, 0_3, 19_4)$,

$(10_1, 0_3, 20_4)$, $(9_1, 0_3, 21_4)$, $(8_1, 0_3, 22_4)$, $(7_1, 0_3, 23_4)$,

$(6_1, 0_3, 24_4)$, $(5_1, 0_3, 25_4)$, $(4_1, 0_3, 26_4)\}$.

All blocks containing ∞ are generated by the blocks of B_1. Also, the blocks of B_1 contain a pair (x_1, y_3) where $x - y \equiv 2 \pmod{27}$ and a pair (x_2, y_4) where $x - y \equiv 11 \pmod{27}$. The blocks of B_2 cover all pure differences of type 3. The block of B_3 covers the pure difference of type 4 equal to 9. The blocks of B_4 cover all pure differences of type 4 in the set $\{1, 2, \ldots, 8, 10, 11, 12, 13\}$ and contain pairs (x_2, y_4) where $(x-y) \pmod{27}$ is in the set $\{a_r, b_r | r = 1, 2, \ldots, 8, 10, 11, \ldots, 13\} = \{1, 3, 4, \ldots, 10, 12, 13\}$ where the a_r and b_r are from the $(A,13)$-system of example 2.1. The blocks of B_5 cover all pure differences of types 1 and 2 in the set $\{1, 2, \ldots, 13\}$ and contain pairs (x_2, y_1) and (x_3, y_2) where $(x - y) \pmod{27}$ is in the set $\{a_r, b_r | r = 1, 2, \ldots, 13\} = \{1, 2, \ldots, 26\}$ where the a_r and b_r are from the $(A,13)$-system of example 2.1. The blocks of B_6 contain a pair (x_1, y_4) with $x - y \equiv 2 \pmod{27}$, a pair (x_2, y_1) with $x - y \equiv 0 \pmod{27}$, a pair (x_3, y_2) with $x - y \equiv 0 \pmod{27}$, a pair (x_4, y_3) with $x - y \equiv 0$

(mod 27), and pairs (x_2, y_4) with $x - y \equiv 2 \pmod{27}$ and $x - y \equiv 0 \pmod{27}$. The blocks of B_7 contain pairs (x_4, y_3) with $(x - y)(\text{mod } 27)$ in the set $\{1, 2, \ldots, 26\}$, pairs (x_1, y_3) with $(x - y)(\text{mod } 27)$ in the set $\{b_{2s+1} - 2s - 1 - r \mid r = 1, 2, \ldots, 26\} = \{0, 1, 3, 4, \ldots, 26\}$ where b_{2s+1} is from the (A,13)-system of example 2.1 ($b_9 = 11$ in this case), and pairs (x_1, y_4) where $(x - y)(\text{mod } 27)$ is in the set $\{b_{2s+1} - 2s - 1 - 2r \mid r = 1, 2, \ldots, 26\} = \{0, 1, 3, 4, \ldots, 26\}$. So all desired differences are covered. In general, all blocks containing ∞ are generated by the blocks of β_1. Also, the blocks of B_1 contain a pair (x_1, y_3) where $x - y \equiv b_{2s+1} - 2s-1 \pmod{6s + 3}$ if $s \equiv 0$ or $1 \pmod 4$ or $x - y \equiv b_{s+1} - 2s+2$ if $s \equiv 2$ or $3 \pmod 4$ and a pair (x_2, y_4) where $x - y \equiv b_{2s+1}$ $(6s + 3)$. The blocks of B_2 cover all pure differences of type 3. The block of B_3 covers the pure differnence of type 4 equal to 2s+1. The blocks of B_4 cover all pure differences of type 4 in the set $\{1, 2, \ldots, 2s, 2s + 2, 2s + 3, \ldots, 3s + 1\}$ and contain pairs (x_2, y_4) where $(x - y)(\text{mod } 6s + 3)$ is in the set $\{a_r, b_r \mid r = 1, 2, \ldots, 2s, 2s+2, 2s+3, \ldots, 3s+1\}$ where the a_r and b_r are from an $(A, 3s + 1)$-system if $s \equiv 0$ or $1 \pmod 4$ or from a $(B, 3s + 1)$-system if $s \equiv 2$ or $3 \pmod 4$. The blocks of B_5 cover all pure differences of types 1 and 2 in the set $\{1, 2, \ldots, 3s+1\}$ and contain pairs (x_2, y_1) and (x_3, y_2) where $(x - y)(\text{mod } 6s + 3)$ is in the set $\{a_r, b_r \mid r = 1, 2, \ldots, 3s + 1\}$ where the a_r and b_r

are from an (A, 3s + 1)-system if $s \equiv 0$ or 1 (mod 4) or from a (B, 3s + 1)-system if $s \equiv 2$ or 3 (mod 4). If $s \equiv 0$ or 1 (mod 4) then the blocks of B_6 contain pairs (x_2, y_1), (x_3, y_2), (x_2, y_4) and (x_4, y_3) where $x - y \equiv 0 \pmod{6s + 3}$ and contain pairs (x_1, y_4) and (x_2, y_4) where $x - y \equiv b_{2s+1} - 2s - 1 \pmod{6s + 3} = a_{2s+1}$. If $s \equiv 2$ or 3 (mod 4) then the blocks of B_6 contain a pair (x_2, y_1) where $x - y \equiv 6s + 2 \pmod{6s + 3}$, a pair (x_1, y_4) where $x - y \equiv b_{2s+1} - 2s \pmod{6s + 3}$, a pair (x_3, y_2) where $x - y \equiv 6s + 2 \pmod{6s + 3}$, a pair (x_4, y_3) where $x - y \equiv 2 \pmod{6s + 3}$, and pairs (x_2, y_4) where $x - y \equiv 4s + 2 + b_{2s+1} \pmod{6s + 3} = a_{2s+1}$ and where $x - y \equiv 6s + 2 \pmod{6s + 3}$. If $s = 0$ or 1 (mod 4) then the blocks of B_7 contain pairs (x_4, y_3) where $(x - y) \pmod{6s + 3}$ is in the set $\{1, 2, \ldots, 6s + 2\}$, pairs (x_1, y_3) where $(x - y) \pmod{6s + 3}$ is in the set $\{b_{2s+1} - 2s - 1 - r \mid r = 1, 2, \ldots, 6s + 2\} = \{0, 1, \ldots, b_{2s+1} - 2s, b_{2s+1} - 2s - 2, \ldots, 6s + 2\}$, and pairs (x_1, y_4) where $(x - y) \pmod{6s + 3}$ is in the set $\{b_{2s+1} - 2s - 1 - 2r \mid r = 1, 2, \ldots, 6s+2\}, = \{0, 1, 2, \ldots, b_{2s+1} - 2s, b_{2s+1} - 2s - 2, \ldots, 6s + 2\}$. If $s \equiv 2$ or 3 (mod 4) then the blocks of B_7 contain pairs (x_4, y_3) where $(x - y) \pmod{6s + 3}$ is in the set $\{2 + r \mid r = 1, 2, \ldots, 6s + 2\} = \{0, 1, 3, 4, \ldots, 6s + 2\}$, pairs (x_1, y_3) where $(x - y) \pmod{6s + 3}$ is in the set $\{b_{2s+1} - 2s + 2 - r \mid r = 1, 2, \ldots, 6s + 2\} = \{0, 1, \ldots, b_{2s+1} - 2s+1, b_{2s+1} - 2s+3, \ldots, 6s + 2\}$, and pairs (x_1, y_4) where $(x - y) \pmod{6s + 3}$ is in the set $\{b_{2s+1} - 2s - 2r \mid$

$r = 1, 2, \ldots, 6s + 2\} = \{0, 1, \ldots, b_{2s+1} - 2s-1, b_{2s+1} - 2s + 1, \ldots, 6s + 2\}$.

<u>Theorem 4.18</u>. If $v \equiv 21 \pmod{24}$ then there exists a 4-rotational STS(v).

<u>Proof</u>: Let $v = 24s + 21$, $s \geq 0$, let $X = \{\infty, 0_1, 1_1, \ldots, (6s + 4)_1, 0_2, 1_2, \ldots, (6s + 4)_2, 0_3, 1_3, \ldots, (6s + 4)_3, 0_4, 1_4, \ldots, (6s + 4)_4\}$ and let $\alpha = (\infty)(0_1, 1_1, \ldots, (6s + 4)_1)(0_2, 1_2, \ldots, (6s + 4)_2)(0_3, 1_3, \ldots, (6s + 4)_3)(0_4, 1_4, \ldots, (6s+4)_4)$ be a permutation of X. Consider the set of blocks $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$ where B_i is as defined below.

<u>Case 1</u>. $s \equiv 0 \pmod 4$.

$B_1 = \{(\infty, 0_1, (6s+2)_4), (\infty, 0_2, (6s + 4)_3\}$

$B_2 = \{(0_1, r_1, (b_r)_2), (0_2, r_2, (b_r)_3), (0_4, r_4, (b_r)_2) \mid r = 1, 2, \ldots, 3s + 2\}$ where (a_r, b_r), $r = 1, 2, \ldots, 3s + 2$ is a $(B, 3s + 2)$-system,

$B_3 = \{(0_1, (6s + 4)_2, 0_4)\}$

$B_4 = \{(0_3, r_3, (b_r + s)_3 \mid r = 1, 2, \ldots, s\}$ where (a_r, b_r), $r = 1, 2, \ldots, s$ is an (A, s)-system,

$B_5 = \{(0_3, (3s +1)_3, 0_1), (0_3, (3s + 2)_3, (6s + 4)_4)\}$

$B_6 = \{(0_1, (3s + 2)_3, (6s + 3)_4), (0_1, (6s + 4)_3, (6s + 4)_4), (0_1, r_3, (2r)_4) \mid r = 1, 2, \ldots, 3s, 3s + 3, \ldots, 6s + 3\}$.

<u>Case 2</u>. $s \equiv 1 \pmod 4$.

$B_1 = \{(\infty, 0_1, (6s + 2)_4), (\infty, 0_2, 0_3)\}$

$B_2 = \{(0_1, r_1, (b_r)_2), (0_2, r_2, (b_r)_3), (0_4, r_4, (b_r)_2) \mid r = 1, 2, \ldots, 3s + 2\}$ where (a_r, b_r), $r = 1, 2, \ldots, 3s + 2$ is an $(A, 3s + 2)$-system,

$B_3 = \{(0_1, 0_2, 0_4)\}$

B_4, B_5, and B_6 are as in case 1.

<u>Case 3</u>. $s \equiv 2$ or $3 \pmod 4$.

$B_1 = \{(\infty, 0_1, (6s)_4), (\infty, 0_2, (3s + 3)_3)\}$

$B_2 = \{(0_1, r_1, (b_r)_2), (0_2, r_2, (b_r)_3), (0_4, r_4, (b_r)_2), \mid r = 1, 2, \ldots, 3s + 2\}$ where (a_r, b_r), $r = 1, 2, \ldots, 3s + 2$ is a $(C, 3s + 2)$-system.

$B_3 = \{(0_1, (3s + 3)_2, 0_4)\}$

$B_4 = \{(0_3, r_3, (b_r + s)_3) \mid r = 1, 2, \ldots, s\}$ where (a_r, b_r), $r = 1, 2, \ldots, s$ is a (B, s)-system.

$B_5 = \{(0_3, (3s)_3, (3s + 1)_1), (0_3, (3s + 2)_3, (6s + 2)_4)\}$

$B_6 = \{(0_1, 0_3, (6s + 3)_4), (0_1, 1_3, (6s + 2)_4), (0_1, (3s + 3)_3, (6s + 4)_4), (0_1, (r + 1)_3, (2r - 1)_4) \mid r = 1, 2, \ldots, 3s + 1, 3s + 4, \ldots, 6s + 2\}$.

In each case, the blocks of B are the base blocks for a 4-rotational STS(v) under α.

<u>Example 4.9</u>. Suppose $v = 117$ and $s = 4$. Then according to Theorem 4.18, a 4-rotational STS(117) can be constructed on the set $\{\infty, 0_1, 1_1, \ldots, 28_1, 0_2, 1_2, \ldots, 28_2, 0_3, 1_3, \ldots, 28_3, 0_4, 1_4, \ldots, 28_4\}$. Theorem 4.18 yields the following collections of base blocks: $B_1 = \{(\infty, 0_1, 26_4), (\infty, 0_2, 28_3)\}$; $B_2 = \{(0_1, 1_1, 25_2), (0_1, 2_1, 8_2),$ $(0_1, 3_1, 22_2), (0_1, 4_1, 9_2), (0_1, 5_1, 23_2), (0_1, 6_1, 10_2),$ $(0_1, 7_1, 21_2), (0_1, 8_1, 11_2), (0_1, 9_1, 26_2), (0_1, 10_1, 12_2),$ $(0_1, 11_1, 27_2), (0_1, 12_1, 13_2), (0_1, 13_1, 20_2), (0_1, 14_1, 29_2),$ $(0_2, 1_2, 25_3), (0_2, 2_2, 8_3), (0_2, 3_2, 22_3), (0_2, 4_2, 9_3),$ $(0_2, 5_2, 23_3), (0_2, 6_2, 10_3), (0_2, 7_2, 21_3), (0_2, 8_2, 11_3),$ $(0_2, 9_2, 26_3), (0_2, 10_2, 12_3), (0_2, 11_2, 27_3), (0_2, 12_2, 13_3),$ $(0_2, 13_2, 20_3), (0_2, 14_2, 29_3), (0_4, 1_4, 25_2), (0_4, 2_4, 8_2),$ $(0_4, 3_4, 22_2), (0_4, 4_4, 9_2), (0_4, 5_4, 23_2), (0_4, 6_4, 10_2),$ $(0_4, 7_4, 21_2), (0_4, 8_4, 11_2), (0_4, 9_4, 26_2), (0_4, 10_4, 12_2),$ $(0_4, 11_4, 27_2), (0_4, 12_4, 13_2), (0_4, 13_4, 20_2), (0_4, 14_4, 29_2)\}$; $B_3 = \{(0_1, 28_2, 0_4)\}$; $B_4 = \{(0_3, 1_3, 6_3), (0_3, 2_3, 11_3), (0_3, 3_3, 10_3),$ $(0_3, 4_3, 12_3)\}$; $B_5 = \{(0_3, 13_3, 0_1), (0_3, 14_3, 28_4)\}$; $B_6 =$ $\{(0_1, 14_3, 27_4), (0_1, 28_3, 28_4), (0_1, 1_3, 2_4), (0_1, 2_3, 4_4),$ $(0_1, 3_3, 6_4), (0_1, 4_3, 8_4), (0_1, 5_3, 10_4), (0_1, 6_3, 12_4), (0_1, 7_3, 14_4),$ $(0_1, 8_3, 16_4), (0_1, 9_3, 18_4), (0_1, 10_3, 20_4), (0_1, 11_3, 22_4),$ $(0_1, 12_3, 24_4), (0_1, 15_3, 1_4), (0_1, 16_3, 3_4), (0_1, 17_3, 5_4),$ $(0_1, 18_3, 7_4), (0_1, 19_3, 9_4), (0_1, 20_3, 11_4), (0_1, 21_3, 13_4),$ $(0_1, 22_3, 15_4), (0_1, 23_3, 17_4), (0_1, 24_3, 19_4), (0_1, 25_3, 21_4),$

$(0_1, 26_3, 23_4), (0_1, 27_3, 25_4)\}$.

All blocks containing ∞ are generated by the blocks of B_1. Also the blocks of B_1 contain a pair (x_4, y_1) where $x - y \equiv 26 \pmod{29}$ and a pair (x_3, y_2) where $x - y \equiv 28 \pmod{29}$. The blocks of B_2 cover all pure differences of types 1, 2, and 4 in the set $\{1, 2, ..., 14\}$ and contain pairs (x_2, y_1), (x_3, y_2), and (x_2, y_4) where $(x - y) \pmod{29}$ is in the set $\{a_r, b_r \mid r = 1, 2, ..., 14\} = \{0, 1, ..., 27\}$ where the a_r and b_r are from the (B,14)-system of example 2.2. The block of B_3 contains a pair (x_2, y_1) where $x - y \equiv 28 \pmod{29}$, a pair (x_4, y_1) where $x - y \equiv 0 \pmod{29}$, and a pair (x_2, y_4) where $x - y \equiv 28 \pmod{29}$. The blocks of B_4 cover all pure differences of type 3 in the set $\{1, 2, 3, 4\} \cup \{a_r + 4, b_r + 4 \mid r = 1, 2, 3, 4\} = \{1, 2, ..., 12\}$ where the a_r and b_r are based on an (A,4)-system. The blocks of B_5 cover the pure differences of type 3 equal to 13 and 14, contain pairs (x_3, y_1) where $x - y \equiv 0 \pmod{29}$ and $x - y \equiv 13 \pmod{29}$, and contain pairs (x_4, y_3) where $x - y \equiv 14 \pmod{29}$ and $x - y \equiv 28 \pmod{29}$. The blocks of B_6 contain pairs (x_3, y_1) where $(x - y) \pmod{29}$ is in the set $\{1, 2, ..., 12, 14, 15, ..., 28\}$, pairs (x_4, y_1) where $(x - y) \pmod{29}$ is in the set $\{1, 2, ..., 25, 27, 28\}$, and pairs (x_4, y_3) where $(x - y) \pmod{29}$ is in the set $\{0, 1, ..., 13, 15, 16, ..., 27\}$.

If $s \equiv 0 \pmod 4$ then, in general, all blocks containing ∞ are generated by the blocks of B_1. Also, the blocks of B_1 contain a pair (x_4, y_1) where $x - y \equiv 6s + 2 \pmod{6s + 5}$ and a pair (x_3, y_2) where $x - y \equiv 6s + 4 \pmod{6s + 5}$. The blocks of B_2 cover all pure differences of types 1, 2, and 4 in the set $\{1, 2, \ldots, 3s + 2\}$ and contain pairs (x_2, y_1), (x_3, y_2), and (x_2, y_4) where $(x - y) \pmod{6s + 5}$ is in the set $\{a_r, b_r | r = 1, 2, \ldots, 3s + 2\} = \{0, 1, \ldots, 6s + 3\}$ where the a_r and b_r are from a $(B, 3s + 2)$-system. The block of B_3 contains pairs (x_2, y_1) and (x_2, y_4) where $x - y \equiv 6s - 4 \pmod{6s + 5}$, and a pair (x_4, y_1) where $x - y \equiv 0 \pmod{6s + 5}$. The blocks of B_4 cover all pure differences of type 3 in the set $\{1, 2, \ldots, s\} \cup \{a_r + s, b_r + s | r = 1, 2, \ldots, s\} = \{1, 2, \ldots, 3s\}$ where the a_r and b_r are based on an (A, s)-system. The blocks of B_5 cover the pure differences of type 3 equal to $3s + 1$ and $3s + 2$, contain pairs (x_3, y_1) where $x - y \equiv 0 \pmod{6s + 5}$ and $x - y \equiv 3s + 1 \pmod{6s + 5}$ and contain pairs (x_4, y_3) where $x - y \equiv 3s + 2 \pmod{6s + 5}$ and $x - y \equiv 6s + 4 \pmod{6s + 5}$. The blocks of B_6 contain pairs (x_3, y_1) where $(x - y) \pmod{6s + 5}$ is in the set $\{1, 2, \ldots, 3s, 3s + 2, \ldots, 6s + 4\}$, pairs (x_4, y_1) where $(x - y) \pmod{6s + 5}$ is in the set $\{1, 2, \ldots, 6s + 1, 6s + 3, 6s + 4\}$, and pairs (x_4, y_3) where $(x - y) \pmod{6s + 5}$ is in the set $\{0, 1, 2, \ldots, 3s + 1, 3s + 3, 3s + 4, \ldots, 6s + 3\}$. So all desired differences are covered.

If $s \equiv 1 \pmod 4$ then, in general, all blocks containing ∞ are generated by the blocks of B_1. Also the blocks of B_1 contain a pair (x_4, y_1) where $x - y \equiv 6s + 2 \pmod{6s + 5}$ and a pair (x_3, y_2) where $x - y \equiv 0 \pmod{6s + 5}$. The blocks of B_2 cover all pure differences of types 1, 2, and 4 in the set $\{1, 2, \ldots, 3s + 2\}$ and contains pairs (x_2, y_1), (x_3, y_2), and (x_2, y_4) where $(x - y) \pmod{6s + 5}$ is in the set $\{a_r, b_r \mid r = 1, 2, \ldots, 3s + 2\} = \{1, 2, \ldots, 6s + 4\}$ where the a_r and b_r are from an $(A, 3s + 2)$-system. The block of B_3 contains pairs (x_2, y_1), (x_2, y_4), and (x_4, y_1) where $x - y \equiv 0 \pmod{6s + 5}$. The blocks in B_4, B_5, and B_6 cover the same differences as in the case where $s \equiv 0 \pmod 4$ and so again, all desired differences are covered.

If $s \equiv 2$ or $3 \pmod 4$ then, in general, all blocks containing ∞ are generated by the blocks of B_1. Also, the blocks of B_1 contain a pair (x_4, y_1) where $x - y \equiv 6s \pmod{6s + 5}$ and a pair (x_3, y_2) where $x - y \equiv 3s + 3 \pmod{6s + 5}$. The blocks of B_2 cover all pure differences of types 1, 2, and 4 in the set $\{1, 2, \ldots, 3s + 2\}$ and contain pairs (x_2, y_1), (x_3, y_2), and (x_2, y_4) where $(x - y) \pmod{6s + 5}$ is in the set $\{a_r, b_r \mid r = 1, 2, \ldots, 3s + 2\} = \{0, 1, 2, \ldots, 3s + 2, 3s + 4, 3s + 5, \ldots, 6s + 4\}$ where the a_r and b_r are from a $(C, 3s + 2)$-system. The block of B_3 contains pairs (x_2, y_1) and (x_2, y_4) where $x - y \equiv 3s + 3 \pmod{6s + 5}$, and a pair (x_4, y_1)

where $x - y \equiv 0 \pmod{6s + 5}$. The blocks of B_4 cover all pure differences of type 3 in the set $\{1, 2, \ldots, s\} \cup \{a_r, b_r \mid r = 1, 2, \ldots, s\} = \{1, 2, \ldots, 3s - 1, 3s + 1\}$ where the a_r and b_r are based on a (B,s)-system. The blocks of B_5 cover the pure differences of type 3 equal to $3s$ and $3s + 2$, contain pairs (x_3, y_1) where $x - y \equiv 6s + 4 \pmod{6s + 5}$ and $x - y \equiv 6s + 4 \pmod{6s + 5}$, and contain pairs (x_4, y_3) where $x - y \equiv 3s \pmod{6s + 5}$ and $x - y \equiv 6s + 2 \pmod{6s + 5}$. The blocks of B_6 contain pairs (x_3, y_1) where $(x - y) \pmod{6s + 5}$ is in the set $\{0, 1, \ldots, 6s + 3\}$, pairs (x_4, y_1) where $(x - y) \pmod{6s + 5}$ is in the set $\{1, 2, \ldots, 6s - 1, 6s + 1, 6s + 2, 6s + 3, 6s + 4\}$, and pairs (x_4, y_3) where $(x - y) \pmod{6s + 5}$ is in the set $\{0, 1, \ldots, 3s - 1, 3s + 1, 3s + 2, \ldots, 6s + 1, 6s + 3, 6s + 4\}$. So all desired differences are covered.

So there exists a 4-rotational STS(v) if and only if $v \equiv 1, 9, 13$ or $21 \pmod{24}$, that is, $v \equiv 1$ or $9 \pmod{12}$.

Another result, pertaining to automorphisms of type [1, 1, 0, ..., 0, 1, 0, 0, 0], that is, automorphisms with a fixed point, a transposition, and a cycle of length $v - 3$, is the following:

<u>Theorem 4.19</u>. There exists a STS(v) on $\{\infty, a, b, 0, 1, \ldots, N-1\}$ where $v = N + 3$, admitting $\alpha = (\infty)(a,b)(0, 1, \ldots, N-1)$ as an automorphism if and only if $v = 3$ or $v \equiv 1, 7, 9$ or $15 \pmod{24}$, $v > 1$.

<u>Proof</u>: $v \equiv 1$ or 3 (mod 6) so $N \equiv 0$ or 4 (mod 6) and $N \equiv 0, 4, 6, 10, 12, 16, 18$ or 22 (mod 24). One of the blocks of the desired STS(v) will be $(0, N/2, \infty)$. Another will be $(a, 0, x)$ where $x \equiv 1$ (mod 2). So now, the set $\{0, 1, \ldots, N/2\} - \{x, N/2\}$ must be partitioned into difference triples. So in this set, there must be an even number of odd numbers since each difference triple includes either 0 or 2 odd numbers. From this condition, it follows that $N \not\equiv 0, 10, 16$, or 18 (mod 24), $N > 0$.

Clearly, there is such a STS(3).

In each of the constructions below, add the blocks $(0, N/2, \infty)$, $(a, 0, x)$ and (∞, a, b):

<u>Case 1</u>. If $N = 4$ let $x = 1$. If $N \equiv 4$ (mod 24), $N > 4$, say $N = 24s + 4$, $s \geq 1$, then take the blocks

A: $(0, 4s + 1 + r, 8s - r)$ $r = 0, 1, \ldots, 2s - 1$

B: $(0, 8s + 1 + r, 12s + 1 - r)$ $r = 0, 1, \ldots, 2s - 1$

and let $x = 10s + 1$.

<u>Case 2</u>. If $N = 6$, let $x = 1$ and take the block $(0, 2, 4)$. If $N \equiv 6$ (mod 24), $N > 6$, say $N = 24s + 6$, $s \geq 1$ then take the blocks

A: $(0, 4s + 1 + r, 8s + 1 - r)$ $r = 0, 1, \ldots, 2s - 1$

B: $(0, 4s - 1 - 2r, 12s + 2 - r)$ $r = 0, 1, \ldots, 2s - 1$

C: $(0, 8s + 2, 16s + 4)$

and let $x = 6s + 1$.

Case 3. If $N = 12$, let $x = 5$ and take the blocks $(0, 1, 3)$ and $(0, 4, 8)$. If $N \equiv 12 \pmod{24}$, $N > 12$, say $N = 24s + 12$, $s \geq 1$, then take the blocks

A: $(0, 4s + 2 + r, 8s + 3 - r)$ $r = 0, 1, \ldots, 2s$

B: $(0, 4s - 2r, 12s + 5 - r)$ $r = 0, 1, \ldots, 2s - 1$

C: $(0, 8s + 4, 16s + 8)$

and let $x = 10s + 5$.

Case 4. If $N = 22$, let $x = 9$ and take the blocks $(0, 5, 6)$, $(0, 8, 10)$, and $(0, 4, 7)$. If $N \equiv 22 \pmod{24}$, $N > 22$ say $N = 24s + 22$, $s \geq 1$, then take the blocks

A: $(0, 4s + 4 + r, 8s + 7 - r)$ $r = 0, 1, \ldots, 2s + 1$

B: $(0, 4s + 2 - 2r, 12s + 10 - r)$ $r = 0, 1, \ldots, 2s$

and let $x = 10s + 9$.

In each case, the blocks are the base blocks for a STS(v) under the automorphism α.

Example 4.10. Suppose $N = 76$ and $s = 3$. Then according to Theorem 4.19, a STS(79) can be constructed on the set $\{\infty, a, b, 0, 1, \ldots, 75\}$ admitting the automorphism $\alpha = (\infty)(a,b)(0, 1, \ldots, 75)$. Theorem 4.19 yields the following collections of base blocks: A = (0, 13, 24), (0, 14, 23), (0, 15, 22), (0, 16, 21), (0, 17, 20), (0, 18, 19) which contain the differences in the set $\{1, 3, 5, 7, 9, 11, 13, 14, \ldots, 23, 24\}$; B = (0, 25, 37), (0, 26, 36), (0, 27, 35), (0, 28, 34), (0, 29, 33), (0, 30 32) which contain the differences in the set {2, 4, 6, 8, 10, 12, 25, 26, ..., 30 32, 33, ..., 36, 37}; along with the block (0, 38, ∞) which covers the difference 38 and generates all blocks that contain ∞ except (∞, a, b), the block (a, 0, 31) which contains the difference 31 and generates all blocks that contain either a or b except (∞, a, b), and finally the block (∞, a, b). So these are the base blocks for a STS(79) under α. Similarly, in general, the blocks of A contain the differences in the set {1, 3, 5, ... 4s - 1, 4s + 1, 4s + 2, ..., 8s}, the blocks of B contain the differences in the set {2, 4, 6, ..., 4s, 8s + 1, 8s + 2, ..., 10s, 10s + 2, 10s + 3, ..., 12s + 1}, the block (0, 12s + 2, ∞) contains the difference 12s + 2 and generates all blocks containing ∞ except (∞, a, b), the block (a, 10s + 1, 0) contains the difference 10s + 1 and generates all blocks containing a or b except (∞, a, b), and finally the block (∞, a, b) is added. So these are the base blocks for a STS(24s + 7) under $\alpha = (\infty)(a,b)(0, 1, \ldots, 24s + 3)$.

<u>Example 4.11</u>. Suppose N = 54 and s = 2. Then according to Theorem 4.19, a STS(57) can be constructed on the set {∞, a, b, 0, 1, ..., 53} admitting the automorphism $\alpha = (\infty)(a,b)(0, 1, \ldots, 53)$. Theorem 4.19 yields the following collections of base blocks: A = (0, 9, 17), (0, 10, 16), (0, 11, 15), (0, 12, 14) which contain the differences in the set {2, 4, 6, 8, 9, 10, 11, 12, 14, 15, 16, 17}; B = (0, 7, 26),

(0, 5, 25), (0, 3, 24), (0, 1, 23) which contain the differences in the set {1, 3, 5, 7, 19, 20, ..., 26}; C = (0, 18,, 36) which contains the difference 18; along with the block (0, 27, ∞) which covers the difference 27 and generates all blocks that contain ∞ except (∞, a, b), the block (a, 0, 13) which contains the difference 13 and generates all blocks that contain either a or b except (∞, a, b), and finally the block (∞, a, b). So these are the base blocks for a STS(57) under α. Similarly, in general, the blocks of A contain the differences in the set {2, 4, ..., 4s, 4s + 1, 4s + 2, ..., 6s, 6s + 2, 6s + 3, ..., 8s + 1}, the blocks of B contain the differences in the set {1, 3, ..., 4s - 1, 8s + 3, 8s + 4, ..., 12s + 2}, the block of C contains the difference 8s + 2, the block (0, 12s + 3, ∞) contains the difference 12s + 2 and generates all blocks containing ∞ except (∞, a, b), the block (a, 0, 6s + 1) contains the difference 6s + 1 and generates all blocks containing a or b except (∞, a, b), and finally the block (∞, a, b) is added. So these are the base blocks for a STS(24s + 9) under $\alpha = (\infty)(a,b)(0,1,..., 24s + 5)$.

Example 4.12. Suppose N = 60 and s = 2. Then according to Theorem 4.19, a STS(63) can be constructed on the set {∞, a, b, 0, 1, ..., 62} admitting the automorphism $\alpha = (\infty)(a,b)(0, 1, ..., 62)$. Theorem 4.19 yields the following collections of base blocks: A = (0, 10, 19), (0, 11, 18), (0, 12, 17), (0, 13, 16), (0, 14, 15) which contain the differences in the set {1, 3, 5, 7, 9, 10, ..., 19}; B = (0, 8, 29), (0, 6, 28), (0, 4, 27) which contain the differences in the set {2, 4, 6, 8, 21, 22, 23, 24, 26, 27, 28, 29}; C = (0, 20, 40) which contains the difference 20; along with the block (0, 30, ∞) which covers the difference 30 and generates all blocks that contain ∞ except (∞, a, b), the block (a, 0, 25) which contains the difference 25 and generates all blocks that contain either a or b except (∞, a, b), and finally the block (∞, a, b). So these are the base blocks for a STS(63) under α. Similarly, in general, the blocks of A contain the differences in the set {1, 3, ..., 4s + 1, 4s + 2, ..., 8s + 3}, the blocks of B contain the differences in the set {2, 4, ..., 4s, 8s + 5, 8s + 6, ..., 10s + 4, 10s + 5, 10s + 6, ..., 12s + 5}, the block of C contains the difference 8s + 4, the block (0, 12s + 6, ∞) contains the difference 12s + 6 and generates all blocks ccontaining ∞ except (∞, a, b), the block (a, 0, 10s + 5) contains the difference 10s + 5 and generates all blocks containing a or b except (∞, a, b), and finally the block (∞, a, b) is added. So these are the base blocks for a STS(24s + 15) under $\alpha = (\infty)(a,b)$ (0, 1, ..., 24s + 11).

Example 4.13. Suppose N = 70 and s = 2. Then according to theorem 4.19, a STS(73) can be constructed on the set {∞, a, b, 0, 1, ..., 69} admitting the automorphism $\alpha = (\infty)(a,b)(0, 1, \ldots, 69)$. Theorem 4.19 yields the following collections of base blocks: A = (0, 12, 23), (0, 13, 22), (0, 14, 21), (0, 15, 20), (0, 16, 19), (0, 17, 18) which contain the differences in the set {1, 3, 5, 7, 9, 11, 12, ..., 23}; B

= (0, 10, 34), (0, 8, 33), (0, 6, 32), (0, 4, 31), (0, 2, 30) which contain the differences in the set {2, 4, 6, 8, 10, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34}; along with the block (0, 35, ∞) which covers the difference 35 and generates all blocks that contain ∞ except (∞, a, b), the block (a, 0, 29) which contains the difference 29 and generates all blocks that contain either a or b except (∞, a, b), and finally the block (∞, a, b). So these are the base blocks for a STS(73) under α. Similarly, in general, the blocks of A contain the differences in the set {1, 3, ..., 4s + 3, 4s + 4, ..., 8s + 7}, the blocks of B contain the differences in the set {2, 4, ..., 4s + 2, 8s + 8, 8s + 9, ..., 10s + 8, 10s + 10, 10s + 11, ..., 12s + 10}, the block (0, 12s + 11, ∞) contains the difference 12s + 11 and generates all blocks containing ∞ except (∞, a, b), the block (0, a, 10s + 9) contains the difference 10s + 9 and generates all blocks containing a or b except (∞, a, b), and finally the block (∞, a, b) is added. So these are the blocks for a STS(24s + 25) under α = (∞)(a,b)(0,1, ..., 24s + 21).

Now, with the existence of 1-rotational STS(v) and the STS(v) as described in theorem 4.19, it is easier to establish the existence of a reverse STS(v) for v $\equiv$ 1, 3, or 9 (mod 24). If a 1-rotational STS(v) with v $\equiv$ 3 or 9 (mod 24) exists admitting an automorphism α of type [1, 0, ..., 0, 1, 0], then this same STS(v) is reverse since the automorphism $\alpha^{(v-1)/2}$ is of type [1,(v-1)/2, 0, ..., 0]. Also, if a STS(v) with v $\equiv$ 1 or 9 (mod 24), satisfying theorem 4.19, with the automorphism α of type [1, 1, 0, ..., 0, 1, 0, 0, 0] exists, then this STS(v) is also reverse since $\alpha^{(v-3)/2}$ of type [1, (v-1)/2, 0, ..., 0].

V. STEINER TRIPLE SYSTEMS WITH AN INVOLUTION

In this chapter, the necessary and sufficient conditions for a STS(v) admitting an automorphism π of type $[f, (v-f)/2, 0, \ldots, 0]$ are investigated. This problem can be restricted to $1 < f < v$, since when $f = 1$, this is the reverse STS(v) problem, and when $f = v$, π is the identity automorphism. Throughout, π will be a permutation of $Z_n \times Z_2 \cup Z_f$ and will be equal to $\pi = (1)(2) \ldots (f)(0_0, 0_1)(1_0, 1_1) \ldots ((n-1)_0, (n-1)_1)$, where (x,i) is abbreviated x_i. All results, unless otherwise noted, are due to Hartman and Hoffman [5].

Theorem 5.1. If there exists a STS(v) admitting an automorphism π of type $[f, (v-f)/2, 0, \ldots, 0]$ then $v \equiv 1$ or $3 \pmod 6$, $f \equiv 1$ or $3 \pmod 6$, and ($v - f \equiv 0 \pmod 4$ and $v \geq 2f + 1$) or ($v - f \equiv 2 \pmod 4$ and $v \geq 3f$).

Proof: Certainly, it must be that $v \equiv 1$ or $3 \pmod 6$. Also, from theorem 1.1, it follows that $f \equiv 1$ or $3 \pmod 6$. If a STS(f) is a subsystem of a STS(v) then it must be that $v \geq 2f + 1$ if $v \neq f$.

Now suppose the STS(v) has point set $X = Z_f \cup (Z_n \times Z_2)$ and automorphism π as described above. So $v = 2n + f$. Now consider the set $S = \{(x, y_i, z_j) \mid x \in Z_f\}$. S contains n blocks. When π acts on S, the blocks are permuted in pairs. So if n is odd, there must be at least one block in S that is mapped onto itself by π, say (x, y_0, y_1). Since there are n pairs (y_0, y_1), it follows that $n \geq f$

and $v \geq 3f$ when n is odd. So if n is even, $v - f = 2n \equiv 0 \pmod 4$ and $v \geq 2f + 1$. If n is odd, $v - f = 2n \equiv 2 \pmod 4$ and $v \geq 3f$.

Now for some definitions from graph theory (for undefined terms, see [1]). A 1-factor, or perfect matching, of a graph $G = (V,E)$ is a subset of E which partitions V. A 1-factorization F of G is a partitioning $\{F_1, F_2, \ldots, F_r\}$ of E into 1-factors. An f-automorphism of a 1-factorization is a permutation σ of V which fixes each 1-factor, i.e. $\sigma(F_i) = F_i$ for all $1 \leq i \leq r$. A 1-factorization which admits σ as an f-automorphism will be called a σ-factorization. For $x \in Z_m$, define $|x|$ by

$$|x| = \begin{cases} x & \text{if } 0 \leq x \leq m/2 \\ -x & \text{otherwise.} \end{cases}$$

Let $m \geq 2$ be an integer and let L be a non-empty subset of $\{1, 2, \ldots, [\![m/2]\!]\}$. The cyclic graph $G(m,L)$ is defined to be the graph with vertex set Z_m and edge set E defined by $\{x,y\} \in E$ if and only if $|y - x| \in L$. Given a graph $G = (V,E)$, we say that the graph $H = (V \times Z_2, E_2)$ if a doubling of G if for every edge $\{x,y\} \in E$ precisely one of the following holds:

(i) $\{x_0, y_0\} \in E_2$ and $\{x_1, y_1\} \in E_2$

(ii) $\{x_0, y_1\} \in E_2$ and $\{x_1, y_0\} \in E_2$.

The following theorem is due to Stern and Lenz [13].

Theorem 5.2. A cyclic graph $G(m,L)$ has a 1-factorization if and only if $m/\gcd(i,m)$ is even for some $i \in L$.

Theorem 5.3. If a graph has a 1-factorization, then any doubling H of G has a π-factorization.

Proof: For each 1-factor of G, construct a 1-factor of H by replacing each edge $\{x,y\}$ by the two edges $(\{x_0,y_0\}$ and $\{x_1,y_1\})$ or $(\{x_0,y_1\}$ and $\{x_1,y_0\})$ whichever is appropriate.

The proof of the following theorem is based on ideas of Stern and Lenz.

Theorem 5.4. Let G be a simple regular graph and let H be a graph formed by taking a doubling of G and adding all edges of the form $\{x_0,x_1\}$. Then H has a π-factorization.

Proof: If G is regular of degree r, then by Vizing's theorem [1], G has a proper edge coloring in either r or $r + 1$ colors. Let C_i $0 \leq i \leq r$ be the set of edges receiving color i. Each vertex x in G is incident with edges of r different colors, so there is a color $f(x)$ such that $C_{f(x)}$ contains no edge incident with x. Form the 1-factor F_i of H by replacing each edge $\{x,y\} \in C_i$ by the two edges $(\{x_0,y_0\}$ and $\{x_1,y_1\})$ or $(\{x_0,y_1\}$ and $\{x_1,y_0\})$ whichever is appropriate, and all edges $\{z_0,z_1\}$ where C_i contains no edge incident with z in G, i.e. $f(z) = i$. Then this 1-factorization of H is a π-factorization.

<u>Theorem 5.5</u>. If $v > f > 1$, $v \equiv 1$ or 3 (mod 6), $f \equiv 1$ or 3 (mod 6), $f \neq 1$ and ($v - f \equiv 0$ (mod 4) and $v \geq 2f + 1$) or ($v - f \equiv 2$ (mod 4) and $v \geq 3f$) then there exists a STS(v) admitting an automorphism π of type [f, (v-f)/2, 0, ..., 0].

<u>Proof</u>: Suppose β_0 is a set of triples of $Z_n \times Z_2$. Associate with β_0 the graph $\Gamma(\beta_0)$ with vertex set $Z_n \times Z_2$ and let $\{x_i, y_j\}$ be an edge of Γ if and only if $\{x_i, y_j\}$ is contained in no triple of β_0. Now, if $\Gamma(\beta_0)$ has a π-factorization with f 1-factors, $F_1, F_2, \ldots, F_f$, then the set $\beta = \beta_0 \cup \beta_1 \cup \beta_3$ is the block set for a STS(v) with point set $Z_n \times Z_2 \cup Z_f$ admitting π as an automorphism where β_1 and β_3 are defined as follows. Let β_3 be the block set of a STS(f) with point set Z_f. Let $\beta_1 = \{(k, x_i, y_j) \mid \{x_i, y_j\} \in F_k, k \in Z_f\}$. Notice that a block is in β_i if and only if it contains i elements of Z_f.

In each of the following cases the set β_0 is presented and, unless otherwise stated, $\Gamma(\beta_0)$ has a π-factorization for the following reasons. If n is odd, $\Gamma(\beta_0)$ satisfies the conditions of Theorem 5.4. If n is even then $\Gamma(\beta_0)$ can be expressed as the disjoint union of a graph satisfying the conditions of theorem 5.4 and a second graph which is a doubling of G(n,L) with $n/2 \in L$ since in each of the following constructions, β_0 will not contain pairs

(x_0, y_1) or (x_0, y_0) where the difference associated with x and y is $n/2$ (and so the graph $\Gamma(\beta_0)$ will contain a doubling of the cyclic graph $G(n, n/2)$ as a subgraph). This second graph has a π-factorization by Theorem 5.2 and Theorem 5.3. In each of the cases, σ denotes the permutation of $Z_n \times Z_2$ defined by
$\sigma(x_i) = \sigma(x+1)_i$.

<u>Construction 1</u>. Consider the following families of base blocks:

A_1: $(0_0, (2k+1+i)_0, (4k-1-i)_1)$ $i = 0, 1, \ldots, k-2$

B_1: $(0_0, (4k+i)_1, (6k-i)_1)$ $i = 0, 1, \ldots, k-1$

C_1: $(0_0, (3k)_0, (5k)_1)$

D_1: $(0_0, (2k+1+i)_1, (4k-i)_0)$ $i = 0, 1, \ldots, k-1$

$E_1(\delta)$: $(0_0, (4k+1+\delta+i)_0, (6k+\delta-i)_0)$ $i = 0, 1, \ldots, k-1$

F_1: $(0_0, (4s+1)_0, (8k+2)_0)$

<u>Case 1.1</u>. $n = 12k+1$, $k > 0$, $f = 6t+1$, $0 < t < 2k$.
Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the blocks in B_1, C_1, D_1 and any $2k-1-t$ of the blocks in A_1 and $E_1(0)$.

<u>Case 1.2</u>. $n = 12k+2$, $k \geq 0$, $f = 6t+3$, $0 \leq t \leq 4k$.
Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of any $4k-t$ of the blocks in A_1, B_1, C_1, D_1, and $E_1(0)$.

<u>Case 1.3</u>. $n = 12k+3$, $k > 0$, $f = 6t+3$, $0 \leq t < 2k$.

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the blocks in $E_1(1)$, A_1, C_1, F_1 and $(0_0, (2k+1)_1, (4k)_0) \in D_1$ and any other $2k-1-t$ of the blocks in B_1 and D_1.

<u>Case 1.4</u>. $n = 12k+3$, $k > 0$, $f = 6t+1$, $0 < t < 2k$.

<u>Case 1.4.1</u>. $1 \leq t < 2k-1$.

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the blocks in B_1, C_1, D_1, F_1 and the block $(0_0, (4k+2)_0, (6k+1)_0) \in E_1(1)$ along with any $2k-2-t$ other of the blocks in A_1 and $E_1(1)$ omitting the block $A_1^x = (0_0, (3k-1)_0, (3k+1)_1)$ from A_1. Now add the $\langle\sigma^3,\pi\rangle$ orbit of A_1^x.

<u>Case 1.4.2</u>. $t = 2k-1$.

If $k = 1$, let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the blocks $(0_0, 5_0, 10_0)$, $(0_0, 6_0, 7_0)$, $(0_0, 4_0, 6_1)$, $(0_0, 1_1, 4_1)$ and the $\langle\sigma^3,\pi\rangle$ orbit of the $(0_0, 5_1, 7_1)$.

If $k > 2$, let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the blocks in $E_1(1)$, A_1, C_1, F_1 and $(0_0, (2k+1)_1, (4k)_0) \in D_1$. Also add the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0, 7_1, 11_1)$ if $k = 2$, of $(0_0, 2_0, 16_1)$ if $k = 3$, of $(0_0, 2_0, (3k+4)_0)$ if $k > 3$.

<u>Case 1.5</u>. $n = 12k+4$, $k \geq 0$, $f = 6t+1$, $0 < t \leq 4k+1$.

If $k = 0$, let $\beta_0 = \emptyset$. If $k > 0$, let β_0 be the $\langle\sigma,\pi\rangle$ orbits of any $4k+1-t$ of the blocks in A_1, B_1, C_1, D_1, and $E_1(1)$.

Construction 2. Consider the following families of base blocks:

$A_1(\delta)$: $(0_0, (2k + 1 + \delta + i)_1, (4k + 1 + \delta + i)_1)$ $i = 0, 1, \ldots, k-1$

$B_2(\delta($: $(0_0, (4k + 2 + \delta + i)_1, (6k + 2 + \delta - i)_0)$ $i = 0, 1, \ldots, k-1$

C_2: $(0_0, (3k + 1)_1, (5k + 2_1)$

D_2: $(0_0, (2k + 2 + i)_0, (4k + 1 - i)_0)$ $i = 0, 1, \ldots, k - 1$

E_2: $(0_0, (4k + 3 + i)_0, (6k + 2 - i)_1)$ $i = 0, 1, \ldots, k - 1$

F_2: $(0_0, (4k + 2)_0, (8k + 4)_0)$

G_2: $(0_0, (4k + 2)_0, (6k + 3)_1)$

Case 2.1. $n = 12k + 5$, $k \geq 0$, $f = 6t + 3$, $0 \leq t \leq 2k$.

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the blocks in $B_2(0)$, C_2, D_2 and any $2k - t$ of the blocks in $A_2(0)$ and E_2.

Case 2.2. $n = 12k + 6$, $k \geq 0$, $f = 6t + 3$, $0 \leq t \leq 4k + 1$.

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of F_2 and any $4k + 1 - t$ of the blocks in $A_2(0)$, $B_2(0)$, C_2, D_2, and E_2.

Case 2.3. $n = 12k + 6$, $k \geq 0$, $f = 6t + 1$, $0 < t \leq 4k + 1$.

If $k = 0$, let β_0 be the $\langle\sigma,\pi\rangle$ orbit of F_2 and the $\langle\sigma^3,\pi\rangle$ orbit of C_2. The graph $\Gamma(\beta_0)$ has a π-factorization using the 1-factors

$F_1 = \{\{0_1,1_1\}, \{2_1,3_1\}, \{4_1,5_0\}, \{5_1,4_0\}, \{0_0,1_0\}, \{2_0,3_0\}\}$,

$F_2 = \{\{0_1,5_1\}, \{1_1,2_0\}, \{2_1,1_0\}, \{3_1,4_1\}, \{0_0,5_0\}, \{3_0,4_0\}\}$,

$F_3 = \{\{0_1,5_0\}, \{1_1,3_0\}, \{2_1,4_0\}, \{3_1,1_0\}, \{4_1,2_0\}, \{5_1,0_0\}\}$,

$F_4 = \{\{0_1,4_0\}, \{1_1,5_0\}, \{2_1,3_0\}, \{3_1,2_0\}, \{4_1,0_0\}, \{5_1,1_0\}\}$, and theorem 5.4.

If $k > 0$, let β_0 be the $\langle\sigma,\pi\rangle$ orbit of F_2, the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0, (3k+1)_0, (3k+2)_0) \in D_2$ and any $4k+1-t$ of the $\langle\sigma,\pi\rangle$ orbits of the remaining blocks in $A_2(0)$, $B_2(0)$, C_2, D_2 and E_2.

Case 2.4. $n = 12k+7$, $k > 0$, $f = 6t+1$, $0 < t \leq 2k$.

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of $B_2(1)$, D_2, G_2, $(0_0, (5k+2)_0, (5k+3)_1) \in E_2$ and any $2k-t$ of the remaining blocks in $A_2(1)$ and E_2.

Case 2.5. $n = 12k+8$, $k \geq 0$, $f = 6t+3$, $0 < t \leq 4k+2$.

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of any $4k+2-t$ of the blocks in $A_2(1)$, $B_2(1)$, D_2, E_2 and G_2.

Construction 3. Consider the following families of base blocks:

$A_3(\delta)$: $(0_0, (2k+2+\delta+i)_\delta, (4k+2+\delta-i)_\delta)$ $i = 0, 1, \ldots, k-1$

$B_3(\delta)$: $(0_0, (4k+4+\delta+i)_\delta, (6k+4+\delta-i)_{1-\delta})$ $i = 0, 1, \ldots, k-1$

$C_3(\delta)$: $(0_0, (4k+3+\delta)_0, (8k+6+2\delta)_0)$

$D_3(\delta)$: $(0_0, (3k+2+\delta)_\delta, (5k+4+\delta)_{1-\delta})$

$E_3(\delta)$: $(0_0, (2k+3+i)_{1-\delta}, (4k+2-i)_{1-\delta})$ $i = 0, 1, \ldots, k-1$

$F_3(\delta)$: $(0_0, (4k+3+\delta+i)_{1-\delta}, (6k+4+\delta-i)_\delta)$ $i = \delta, \delta+1, \ldots, k$

G_3: $(0_0, (2k+2)_0, (4k+3)_0)$.

Case 3.1. $n = 12k+9$, $k \geq 0$, $f = 6t+3$, $0 \leq t \leq 2k$.

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of $A_3(0)$, $C_3(0)$, $D_3(0)$, $F_3(0)$ and any $2k-t$ of the blocks in $B_3(0)$ and $E_3(0)$.

<u>Case 3.2</u>. $n = 12k + 9,\ k > 0,\ f = 6t + 1,\ 0 < t \leq 2k.$

<u>Case 3.2.1</u>. $k \equiv 1 \pmod 3$.

Let β_0 be the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0, (5k+3)_0, (5k+5)_1) \in B_3(0)$ and the $\langle\sigma,\pi\rangle$ orbits of the blocks in $A_3(0)$, $C_3(0)$, $D_3(0)$, $F_3(0)$ and any $2k - t$ of the remaining blocks in $B_3(0)$ and $E_3(0)$.

<u>Case 3.2.2</u>. $k \equiv 2 \pmod 3$.

Let β_0 be the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0, (5k+3)_1, (5k+4)_0) \in F_3(0)$ and the $\langle\sigma,\pi\rangle$ orbits of $(0_0, (4k+3)_1, (6k+4)_0) \in F_3(0)$, the blocks in $B_3(0)$, $C_3(0)$, $D_3(0)$, $E_3(0)$ and any $2k - t$ of the remaining blocks in $A_3(0)$ and $F_3(0)$.

<u>Case 3.2.3</u>. $k \equiv 0 \pmod 3$.

Let β_0 be the $\langle\sigma^3,\pi\rangle$ orbit of $D_3(0)$ and the $\langle\sigma,\pi\rangle$ orbits of $(0_0, (3k+2)_1, (3k+3)_1) \in E_3(0)$ the blocks in $A_3(0)$, $C_3(0)$, $F_3(0)$ and any $2k - t$ of the remaining blocks in $B_3(0)$ and $E_3(0)$.

<u>Case 3.3</u>. $n = 12k + 10,\ k \geq 0,\ f = 6t + 1,\ 0 < t \leq 4k + 3.$

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of any $4k + 3 - t$ of the blocks in $A_3(0)$, $B_3(0)$, $D_3(0)$, $E_3(0)$ and $F_3(0)$.

<u>Case 3.4</u>. $n = 12k + 11,\ k \geq 0,\ f = 6t + 3,\ 0 \leq t \leq 2k + 1.$

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of $(0_0, (4k+4)_0, (6k+5)_1)$, the blocks in $A_3(1)$, $D_3(1)$, $F_3(1)$ and any $2k+1-t$ of the blocks in $B_3(1)$, $E_3(1)$, and G_3.

Case 3.5. $n = 12k + 12$, $k \geq 0$, $f = 6t + 3$, $0 < t \leq 4k + 3$.
Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of $C_3(1)$ and any $4k + 3 - t$ of the blocks in $A_3(1)$, $B_3(1)$, $D_3(1)$, $E_3(1)$, $F_3(1)$, and G_3.

Case 3.6. $n = 12k + 12$, $k \geq 0$, $f = 6t + 1$, $1 < t \leq 4k + 3$.
If $k = 0$, let β_0 be the $\langle\sigma^3,\pi\rangle$ orbits of $(0_0, 4_0, 8_0)$, $(0_0, 1_0, 5_1)$ and the $\langle\sigma,\pi\rangle$ orbit of $(0_0,3_0,5_0)$ (omit if $t = 1$). If $k > 0$, let β_0 be the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0,(3k+2)_1, (3k+4)_1) \in A_3(1)$, and the $\langle\sigma,\pi\rangle$ orbits of $C_3(1)$ and any $4k + 3 - t$ of the remaining blocks in $A_3(1)$, $B_3(1)$, $D_3(1)$, $E_3(1)$, $F_3(1)$, and G_3.

Case 4. $n \equiv 1$ or $3 \pmod 6$, $f = n$.
Let β_0 be the $\langle\pi\rangle$ orbits of the blocks of a STS(n) with point set $Z_n \times \{0\}$.

The constructions for Case 5 are adaptations of Rosa's constructions [10] for cyclic STS(v).

Case 5. $n \equiv 3 \pmod 6$, $f = n - 2$.

Case 5.1. $n = 24k + 3$, $f = 24k + 1$, $k > 0$.
Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the following blocks:

$(0_0, (4k + 1 + i)_0, (8k - i)_0)$ $i = 0,1, \ldots, k - 2$

$(0_0, (8k + 2 + i)_0, (12k - i)_0)$ $i = 0,1, \ldots, k - 2$

$(0_0,\ (5k + i)_0,\ (7k - 1 - i)_0)$ $i = 0,1,\ \ldots,\ k - 2$

$(0_0,\ (9k + 2 + i)_0,\ (11k - i)_0)$ $i = 0,1,\ \ldots,\ k - 2$

$(0_0,\ (7k)_0,\ (9k + 1)_0)$

$(0_0,\ (7k + 1)_0,\ (11k + 1)_0)$

$(0_0,\ (10k + 1)_0,\ (12k + 1)_0)$

$(0_0,\ (8k + 1)_0,\ (16k + 2)_0)$

$(0_0,\ (6k - 1)_0,\ (6k)_1)$

and the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0,4_1,5_1)$.

<u>Case 5.2</u>. $n = 24k - 3,\ f = 24k - 5,\ k > 0$.

Let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the following blocks:

$(0_0,\ (4k + i)_0,\ (8k - 2 - i)_0)$ $i = 0,1,\ \ldots,\ 2k - 2$

$(0_0,\ (8k + 1 + i)_0,\ (12k - 2 - i)_0)$ $i = 0,1,\ \ldots,\ k - 3$

$(0_0,\ (9k + i)_0,\ (11k - 3 - i)_0)$ $i = 0,1,\ \ldots,\ k - 3$

$(0_0,\ (6k - 1)_0,\ (10k - 2)_0)$

$(0_0,\ (11k - 2)_0,\ (11k - 1)_1)$

$(0_0,\ (8k - 1)_0,\ (16k - 2)_0)$

$(0_0,\ (8k)_0,\ (10k - 1)_0)$ (omit if $k = 1$)

$(0_0,\ (9k - 1)_0,\ (11k)_0)$ (omit if $k = 1$)

and the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0,4_1,5_1)$.

<u>Case 5.3</u>. $n = 24k + 9,\ f = 24k + 7,\ k \geq 0$.

If $k = 0$, let β_0 be the $\langle\sigma,\pi\rangle$ orbit of $(0_0, 3_0, 6_0)$ and the $\langle\sigma^3,\pi\rangle$ orbits of $(0_0, 1_0, 2_0)$, $(0_0, 5_0, 7_0)$, $(0_0, 4_0, 8_1)$ and $(0_0, 1_1, 5_1)$.
If $k = 1$ let β_0 be the $\langle\sigma,\pi\rangle$ orbits of $(0_0, 6_0, 10_0)$, $(0_0, 7_0, 12_0)$, $(0_0, 13_0, 15_0)$, $(0_0, 14_0, 17_0)$, $(0_0, 11_0, 22_0)$, $(0_0, 8_0, 9_1)$, and the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0, 4_1, 5_1)$.
If $k > 1$, let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the following blocks:

$(0_0, (4k + 2 + i)_0, (8k + 2 - i)_0)$ $i = 0,1, \ldots, 2k - 1$

$(0_0, (8k + 4 + i)_0, (12k + 3 - i)_0)$ $i = 0, 1, \ldots, k - 3$

$(0_0, (9k + 2 + i)_0, (11k + 3 - i)_0)$ $i = 0, 1, \ldots, k - 1$

$(0_0, (6k + 2)_0, (10k + 3)_0)$

$(0_0, (10k + 2)_0, (12k + 5)_1)$

$(0_0, (8k + 3)_0)$ $(16k + 6)_0)$

$(0_0, (11k + 4)_0, (11k + 5)_1)$

and the $\langle\sigma,\pi\rangle$ orbit of $(0_0, 4_1, 5_1)$.

<u>Case 5.4</u>. $n = 24k + 15$, $f = 24k + 13$, $k \geq 0$

If $k = 0$, let β_0 be the $\langle\sigma,\pi\rangle$ orbits of $(0_0, 2_0, 8_0)$, $(0_0, 5_0, 10_0)$, $(0_0, 1_1, 4_0)$, and the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0, 4_1, 5_1)$. If $k > 0$, let β_0 be the $\langle\sigma,\pi\rangle$ orbits of the following blocks:

$(0_0, (4k + 3 + i)_0, (8k + 4 - i)_0)$ $i = 0,1, \ldots, 2k - 1$

$(0_0, (8k + 7 + i)_0, (12k + 5 - i)_0)$ $i = 0,1, \ldots, k - 2$

$(0_0, (9k + 6 + i)_0, (11k + 4 - i)_0)$ $i = 0,1, \ldots, k - 3$

$(0_0, (6k + 3 + i)_0, (10k + 5 - i)_0)$ $i = 0,1$

$(0_0,\ (8k + 6)_0,\ (10k = 6)_0)$ (omit if $k = 1$)

$(0_0,\ (12k + 6)_0,\ (12k + 8)_0)$

$(0_0,\ (8k + 5)_0,\ (16k + 10)_0)$

$(0_0,\ (11k + 5)_0,\ (11k + 6)_1)$

and the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0,\ 4_1,\ 5_1)$.

Case 6. $n = 12k + 8$, $f = 3$, $k \geq 0$.

Consider the following blocks:

$(0_0,\ (4k + 3 + i)_\delta, (6k + 4 - i)_\delta)$ $i = 0, 1, \ldots, k$

$(0_0,\ (4k + 3 + i)_{i - \delta},\ (6k + 3 - i)_{1 - \delta})$ $i = 0, 1, \ldots, k - 1$

$(0_0,\ (3k + 2 + i)_{1 - \delta},\ (3k + 1 - i)_\delta)$ $i = 0, 1, \ldots, k - 1$

$(0_0,\ (3k + 3 + i)_\delta,\ (3k + 1 - i)_{1 - \delta})$ $i = 0, 1, \ldots, k - 1$

$(0_0,\ (3k + 2)_\delta,\ (5k + 3)_{1 - \delta})$.

Let β' be the images of these blocks under the transformations σ^j where $\delta = 0$ when $0 \leq j < 6k + 4$ and $\delta = 1$ when $6k + 4 \leq j < 12k + 8$. Now let β_0 be the $\langle\pi\rangle$ images of the blocks in β'. the graph $\Gamma(\beta_0)$ has the following edges: the $\langle\sigma\rangle$ orbit of $\{0_0, 0_1\}$ and the $\langle\pi\rangle$ orbits of the images of $\{0_0,\ (4k - 2)_{1 - \delta}\}$ under the transformations σ^j defined above. This graph has a π-factorization by theorem 5.4.

Case 7. $n = 12k$, $f = 3$ or 7, $k > 0$.

Consider the following blocks:

$(0_0, (4k + 1 + i)_\delta, (6k - i)_{1-\delta})$ $i = 0, 1, \ldots, k - 1$

$(0_0, (4k + 1 + i)_{1-\delta}, (6k - 1 - i)_\delta)$ $i = 0, 1, \ldots, k - 2$

$(0_0, (2k)_{1-\delta}, (5k)_{1-\delta})$

Let β' be the images of these blocks under the transformations σ^j where $\delta = 0$ when $0 \leq j < 6k + 4$ and $\delta = 1$ when $6k + 4 \leq j < 12k + 8$. Now let β_0 be the $\langle\pi\rangle$ images of the blocks in β' and the $\langle\sigma,\pi\rangle$ orbits of the following blocks

$(0_0, (2k + 1 + i)_0, (4k - 1 - i)_0)$ $i = 0, 1, \ldots, k - 2$

$(0_0, (2k + 1 + i)_1, (4k - i)_1)$ $i = 0, 1, \ldots, k - 1$

$(0_0, (4k)_0, (8k)_0)$.

The graph $\Gamma(\beta_0)$ contains the images of $\{0_0, 0_1\}$ and $\{0_0, (2k)_\delta\}$ and the existence of π-factorization follows from theorem 5.4. This covers the case when $f = 3$. When $f = 7$ and $k > 1$, take the $\langle\sigma^3,\pi\rangle$ orbit of $(0_0, (3k - 1)_0, (3k + 1)_0)$ instead of the $\langle\sigma,\pi\rangle$ orbit of it. When $f = 7$ and $k = 1$, let β_0 be the $\langle\sigma,\pi\rangle$ orbit of $(0_0, 4_0, 7_1)$, $(0_0, 3_0, 5_0)$, and the $\langle\pi\rangle$ orbits of the σ^j images of $(0_0, 1_0, 2_1)$ with $j = 0, 1, 3, 4, 6, 7, 9, 10$. So, there exists a STS (v) admitting an automorphism π of type $[f, (v - f)/2, 0, \ldots, 0]$ if and only if $v \equiv 1$ or $3 \pmod 6$, $f \equiv 1$ or $3 \pmod 6$, and ($v - f \equiv 0 \pmod 4$ and $v \geq 2f + 1$) or ($v - f \equiv 2 \pmod 4$ and $v \geq 3f$).

Example 5.1. Suppose $n = 6$ and $f = 7$. Then according to theorem 5.5, a STS(19) can be constructed on the set $Z_6 \times Z_2 \cup Z_7$ admitting the automorphism $\pi = (1)\ (2)\ (3)\ (4)\ (5)\ (6)\ (7)\ (0_0,0_1)\ (1_0,1_1) \cdots (6_0,6_1)$. Case 2.3 produces the following collection of blocks:

$\beta_0 = \{(0_0, 2_0, 4_0), (1_0, 3_0, 5_0), (0_0, 1_1, 2_1), (0_1, 2_1, 4_1), (1_1, 3_1, 5_1), (0_1, 1_0, 2_0)\}$. These blocks produce a graph $\Gamma(\beta_0)$ with edges as listed in the sets F_1, F_2, F_3 and F_4 in case 2.3 along with the edges in the sets $F_5 = \{\{0_0, 0_0\}, \{1_0, 1_1\} \{2_0, 2_1\}, \{3_0, 3_1\} \{4_0, 4_1\}, \{5_0, 5_1\}\}$, $F_6 = \{\{0_0, 3_1\}, \{1_0, 4_1\}, \{2_0, 5_1\}, \{3_0, 0_1\} \{4_0, 1_1\}, \{5_0, 2_1\}\}$, and $F_7 = \{\{0_0, 3_0\}, \{1_0, 4_0\} \{2_0, 5_0\}, \{0_1, 3_1\}, \{1_1, 4_1\}, \{2_1, 5_1\}\}$. The sets F_1, F_2, F_3, F_4, F_5, F_6, and F_7 are 1-factors for the graph $\Gamma(\beta_0)$. Also, these 1-factors are fixed under π, so they form a π-factorization of $\Gamma(\beta_0)$. The blocks of β_1 are as follows: $\beta_1 = \{(1, 0_1, 1_1), (1, 2_1, 3_1), (1, 4_1, 5_0), (1, 5_1, 4_0), (1, 0_0, 1_0), (1, 2_0, 3_0), (2, 0_1, 5_1), (2, 1_1, 2_0), (2, 2_1, 1_0), (2, 3_1, 4_1), (2, 0_0, 5_0), (2, 3_0, 4_0), (3, 0_1, 5_0), (3, 1_1, 3_0), (3, 2_1, 4_0), (3, 3_1, 1_0), (3, 4_1, 2_0), (3, 5_1, 0_0), (4, 0_1, 4_0), (4, 1_1, 5_0), (4, 2_1, 3_0), (4, 3_1, 2_0), (4, 4_1, 0_0), (4, 5_1, 1_0), (5, 0_0, 0_1), (5, 1_0, 1_1), (5, 2_0, 2_1), (5, 3_0, 3_1), (5, 4_0, 4_1), (5, 5_0, 5_1), (6, 0_0, 3_1), (6, 1_0, 4_1), (6, 2_0, 5_1), (6, 3_0, 0_1), (6, 4_0, 1_1), (6, 5_0, 2_1), (7, 0_0, 3_0), (7, 1_0, 4_0), (7, 2_0, 5_0), (7, 0_1, 3_1), (7, 1_1, 4_1), (7, 2_1, 5_1)\}$. Now, the blocks of β_3 are as follows:

$\beta_3 = \{(1, 3, 4), (2, 4, 5), (3, 5, 6), (4, 6, 7), (5, 7, 0), (6, 0, 1) (7, 1, 2)\}$. The blocks of β_0 contain certain pairs of the form

(x_i, y_j) and miss others. The missing pairs are contained by blocks of β_1 and, since these blocks are constructed using 1- factors of $\Gamma(\beta_0)$, every pair of type (x_i, y) where $y \varepsilon Z_f$ is also contained here. The blocks of β_3 contain all pairs (x, y) where $x \varepsilon Z_f$ and $y \varepsilon Z_f$. So all desired pairs are present and the blocks of $\beta_0 \cup \beta_1 \cup \beta_3$ are the blocks of a STS(19) and, due to construction, this system admits the automorphism π. Similarly, in general the blocks of β_0 contain certain pairs of the form (x_i, x_j) and miss others. The missing pairs are contained in the blocks of β_1 and, since these blocks are constructed using 1-factors of $\Gamma(\beta_0)$, every pair of type (x_i,y) where $y \varepsilon Z_f$ is also contained here. The blocks of β_3 contain all pairs (x, y) where $x \varepsilon Z_f$ and $y \varepsilon Z_f$. So the blocks of $\beta_0 \cup \beta_1 \cup \beta_3$ are the blocks of a STS (2n + f) admitting the desired automorphism.

Also: L. Teirlinck, A simplification of the proof of the existence of reverse Steiner triple systems of order ≡ 1 (mod 24), Discrete Math 13 (1975) 297-298

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