A Type of Eneström-Kakeya Theorem for Quaternionic Polynomials Involving Monotonicity with a Reversal

Abstract. The Eneström-Kakeya Theorem states that if $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ is a polynomial of degree n with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all zeros of P lie in $|z| \leq 1$ in the complex plane. Motivated by recent results concerning an Eneström-Kakeya "type" condition on the real and imaginary parts of complex coefficients, we give similar results with hypotheses concerning the real and imaginary parts of the coefficients of a quaternionic polynomial. We give bounds on the moduli of quaternionic zeros of such polynomials.

1 Introduction

The classical Eneström-Kakeya Theorem concerns the location of the complex zeros of a real polynomial with nonnegative monotone coefficients. It was independently proved by Gustav Eneström in 1893 [4] and Sōichi Kakeya in 1912 [8].

Theorem 1.1. Eneström-Kakeya Theorem. *If $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of P lie in $|z| \leq 1$.*

A corollary to the main theorem in [6] concerns monotonicity of the real and imaginary parts of the coefficients of a polynomial. The monotonicity condition involves a reversal, as follows.

Theorem 1.2. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients where $Re(a_\ell) = \alpha_\ell$ and $Im(a_\ell) = \beta_\ell$ for $\ell = 0, 1, \ldots, n$. Suppose that $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_n$ and $\beta_0 \leq \beta_1 \cdots \leq \beta_r \geq \beta_{r+1} \geq \cdots \geq \beta_n$. Then all the zeros of P lie in*

$$\min\left\{|a_0|/\left(2(\alpha_k + \beta_r) - (\alpha_0 + \beta_0) - (\alpha_n + \beta_n - |a_n|)\right), 1\right\} \leq |z|$$

$$\leq \max\left\{\left(|a_0| - (\alpha_0 + \beta_0) - (\alpha_n + \beta_n) + 2(\alpha_k + \beta_r)\right)/|a_n|, 1\right\}.$$

Mathematics Subject Classification (2010): 12D10, 30C15, 30E10

Keywords: Location of zeros of a polynomial, quaternionic polynomial, monotone coefficients

By combining more general monotonicity conditions of Aziz and Zargar [1] and Shah et al. [11], the authors of this work recently proved the following [5, Theorem 5].

Theorem 1.3. *Let* $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *be a polynomial of degree* n *with complex coefficients. Let* $\alpha_\ell = Re(a_\ell)$ *and* $\beta_\ell = Im(a_\ell)$ *for* $0 \leq \ell \leq n$. *Suppose that, for some positive numbers* k_R, k_I, ρ_R, ρ_I, p, *and* q *with* $k_R \geq 1$, $k_I \geq 1$, $0 < \rho_R \leq 1$, $0 < \rho_I \leq 1$, *and* $0 \leq q \leq p \leq n$, *the coefficients satisfy*

$$\rho_R \alpha_q \leq \alpha_{q+1} \leq \alpha_{q+2} \leq \cdots \leq \alpha_{p-1} \leq k_R \alpha_p \text{ and}$$

$$\rho_I \beta_q \leq \beta_{q+1} \leq \beta_{q+2} \leq \cdots \leq \beta_{p-1} \leq k_I \beta_p.$$

Then, all the zeros of P *lie in the closed annulus*

$$\min\left\{1, \frac{|a_0|}{M - |a_0| + |a_n|}\right\} \leq |q| \leq \frac{M}{|a_n|},$$

where

$$M = |a_0| + M_r + (1-\rho_R)|\alpha_q| - \rho_R \alpha_q + (1-\rho_I)|\beta_q|$$

$$-\rho_I \beta_q + (k_R - 1)|\alpha_p| + k_R \alpha_p + (k_I - 1)|\beta_p| + k_I \beta_p + M_p,$$

$M_r = \sum_{\ell=1}^{r} |a_\ell - a_{\ell-1}|$, *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

The quaternions, $\mathbb{H} = \{\alpha + \beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$, where $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$, are the standard example of a noncommutative division ring. The modulus of $q \in \mathbb{H}$ is $|q| = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The absence of commutivity leads to some surprising behavior of the zeros of a polynomial of a quaternionic variable. For example, the second degree polynomial $q^2 + 1$ has set of zeros $\{\beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k} \mid \beta^2 + \gamma^2 + \delta^2 = 1\}$.

The Eneström-Kakeya Theorem has been extended to polynomials of a quaternionic variable as follows [2].

Theorem 1.4. *If* $p(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ *is a polynomial of degree* n *(where* q *is a quaternionic variable) with real coefficients satisfying* $0 \leq a_0 \leq \cdots \leq a_n$, *then all the zeros of* p *lie in* $|q| \leq 1$.

By giving results on the location of the quaternionic zeros of a polynomial, we include all (finitely many) complex zeros and potentially infinitely many more quaternionic zeros, as illustrated for polynomial q^2+1. The purpose of this paper is to extend Theorem 1.3 to quaternionic polynomials and, in the process, to introduce a reversal in the monotonicity condition on the real and imaginary parts of the quaternionic coefficients.

2 The Results

Theorem 2.1. *Let $P(q)=\sum_{\ell=0}^{n} q^\ell a_\ell$ be a polynomial of degree n with quaternionic coefficients, that is $a_\ell=\alpha_\ell+\beta_\ell i+\gamma_\ell j+\delta_\ell k$, where for positive real $\rho_{R_1},\rho_{R_2},\rho_{I_1},\rho_{I_2},\rho_{J_1},\rho_{J_2},\rho_{K_1}$, ρ_{K_2} each less than or equal to 1 and for k_R,k_I,k_J,k_K each at least 1, we have*

$$\rho_{R_1}\alpha_r \le \alpha_{r+1} \le \cdots \le \alpha_{\eta-1} \le k_R\alpha_\eta \ge \alpha_{\eta+1} \ge \cdots \ge \rho_{R_2}\alpha_p,$$

$$\rho_{I_1}\beta_r \le \beta_{r+1} \le \cdots \le \beta_{\eta-1} \le k_I\beta_\eta \ge \beta_{\eta+1} \ge \cdots \ge \rho_{I_2}\beta_p,$$

$$\rho_{J_1}\gamma_r \le \gamma_{r+1} \le \cdots \le \gamma_{\eta-1} \le k_J\gamma_\eta \ge \gamma_{\eta+1} \ge \cdots \ge \rho_{J_2}\gamma_p, \text{ and}$$

$$\rho_{K_1}\delta_r \le \delta_{r+1} \le \cdots \le \delta_{\eta-1} \le k_K\delta_\eta \ge \delta_{\eta+1} \ge \cdots \ge \rho_{K_2}\delta_p.$$

Then all zeros of $P(q)$ lie in

$$\min\left\{1, \frac{|a_0|}{M-|a_0|+|a_n|}\right\} \le |q| \le \frac{M}{|a_n|},$$

where

$$\begin{aligned} M &= |a_0| + M_r - \rho_{R_1}\alpha_r + |\alpha_r|(1-\rho_{R_1}) + 2|\alpha_\eta|(k_R-1) + 2k_R\alpha_\eta + |\alpha_p|(1-\rho_{R_2}) - \rho_{R_2}\alpha_p \\ &\quad -\rho_{I_1}\beta_r + |\beta_r|(1-\rho_{I_1}) + 2|\beta_\eta|(k_I-1) + 2k_I\beta_\eta + |\beta_p|(1-\rho_{I_2}) - \rho_{I_2}\beta_p \\ &\quad -\rho_{J_1}\gamma_r + |\gamma_r|(1-\rho_{J_1}) + 2|\gamma_\eta|(k_J-1) + 2k_J\gamma_\eta + |\gamma_p|(1-\rho_{J_2}) - \rho_{J_2}\gamma_p \\ &\quad -\rho_{K_1}\delta_r + |\delta_r|(1-\rho_{K_1}) + 2|\delta_\eta|(k_K-1) + 2k_K\delta_\eta + |\delta_p|(1-\rho_{K_2}) - \rho_{K_2}\delta_p + M_p, \end{aligned}$$

$M_r=\sum_{\ell=1}^{r}|a_\ell-a_{\ell-1}|$, and $M_p=\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$.

With $\rho_{R_1}=\rho_{I_1}=\rho_{J_1}=\rho_{K_1}=1$, $k_R=k_I=k_J=k_K=1$, and $\rho_{R_2}=\rho_{I_2}=\rho_{J_2}=\rho_{K_2}=1$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *If $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ is a polynomial of degree n with quaternionic coefficients, that is $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$, satisfying*

$$\alpha_r \leq \alpha_{r+1} \leq \cdots \leq \alpha_{\eta-1} \leq \alpha_\eta \geq \alpha_{\eta+1} \geq \cdots \geq \alpha_p,$$

$$\beta_r \leq \beta_{r+1} \leq \cdots \leq \beta_{\eta-1} \leq \beta_\eta \geq \beta_{\eta+1} \geq \cdots \geq \beta_p,$$

$$\gamma_r \leq \gamma_{r+1} \leq \cdots \leq \gamma_{\eta-1} \leq \gamma_\eta \geq \gamma_{\eta+1} \geq \cdots \geq \gamma_p, \text{ and}$$

$$\delta_r \leq \delta_{r+1} \leq \cdots \leq \delta_{\eta-1} \leq \delta_\eta \geq \delta_{\eta+1} \geq \cdots \geq \delta_p.$$

Then all zeros of $P(q)$ lie in

$$\min\left\{1, \frac{|a_0|}{M - |a_0| + |a_n|}\right\} \leq |q| \leq \frac{M}{|a_n|},$$

where

$$M = |a_0| + M_r - \alpha_r + 2\alpha_\eta - \alpha_p - \beta_r + 2\beta_\eta - \beta_p - \gamma_r + 2\gamma_\eta - \gamma_p - \delta_r + 2\delta_\eta - \delta_p + M_p,$$

$M_r = \sum_{\ell=1}^{r} |a_\ell - a_{\ell-1}|$, and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

With $r = l$ and $\eta = p = n$, Corollary 2.2 reduces to the following.

Corollary 2.3. *If $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ is a polynomial of degree n with quaternionic coefficients, that is $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$, satisfying*

$$\alpha_l \leq \alpha_{l+1} \leq \cdots \leq \alpha_{n-1} \leq \alpha_n, \ \ \beta_l \leq \beta_{l+1} \leq \cdots \leq \beta_{n-1} \leq \beta_n,$$

$$\gamma_l \leq \gamma_{l+1} \leq \cdots \leq \gamma_{n-1} \leq \gamma_n, \text{ and } \delta_l \leq \delta_{l+1} \leq \cdots \leq \delta_{n-1} \leq \delta_n.$$

Then all zeros of $P(q)$ lie in

$$\min\left\{1, \frac{|a_0|}{M - |a_0| + |a_n|}\right\} \leq |q| \leq \frac{M}{|a_n|},$$

where $M = |a_0| + M_l - \alpha_l + \alpha_n - \beta_l + \beta_n - \gamma_l + \gamma_n - \delta_l + \delta_n$ and $M_l = \sum_{\ell=1}^{l} |a_\ell - a_{\ell-1}|$.

Corollary 2.3 is a slight refinement of a result of Tripathi [12, Theorem 3.1]. Corollary 2.3 implies Theorem 9 of [2] when $l = 0$.

In connection with Bernstein inequalities, Chan and Malik [3] (and, independently, Qazi [10]) considered the class of polynomials of a complex variable of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$. Inspired by this, the current authors considered complex polynomials of the form $P(z) = a_0 + \sum_{r}^{p} a_\ell z^\ell + a_n z^n$ in connection to locations of zeros [5]. An additional result follows from Corollary 2.2 by applying it to a quaternionic polynomial of the form $P(q) = a_0 + \sum_{\ell=r}^{p} q^\ell a_\ell + q^n a_n$ (with the coefficients satisfying the hypotheses of Corollary 2.2). This result gives the location of the zeros of P as stated in Corollary 2.2, where $M_r = |a_0| + |a_r|$ and $M_p = |a_p| + |a_n|$.

3 Lemmas

We adopt the standard that polynomials have the indeterminate on the left and the coefficients on the right, so that we have quaternionic polynomials of the form $P_1(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$. With $P_2(q) = \sum_{\ell=0}^{m} q^\ell b_\ell$, we have the *regular product* $(P_1 * P_2)(q) = \sum_{i=0,1,\ldots,n;\, j=0,1,\ldots,m} q^{i+j} a_i b_j$.

Zeros of regular products of quaternionic polynomials behave as follows [9]:

Theorem 3.1. *Let f and g be given quaternionic polynomials. Then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g(f(q_0)^{-1} q_0 f(q_0)) = 0$.*

Gentili and Struppa introduced a Maximum Modulus Theorem for regular functions [7]:

Theorem 3.2. *Let $B = B(0, r)$ be an open ball in $\mathbb{H}$ with center 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is constant on B.*

4 Proof of Theorem 2.1

Proof of Theorem 2.1. Define $f(q)$ with the equation

$$\begin{aligned} P(q) * (1 - q) &= \left(\sum_{\ell=0}^{n} q^\ell a_\ell \right) * (1 - q) \\ &= a_0 + \sum_{\ell=1}^{n} q^\ell (a_\ell - a_{\ell-1}) - q^{n+1} a_n \end{aligned}$$

$$= \quad f(q) - q^{n+1}a_n.$$

By Theorem 3.1, $P(q) * (1-q) = 0$ if and only if either $P(q) = 0$, or $P(q) \neq 0$ implies $1 - P(q)^{-1}qP(q) = 0$. Notice that $1 - P(q)^{-1}qP(q) = 0$ implies $q = 1$. So the only zeros of $P(q) * (1-q)$ are $q = 1$ and the zeros of $P(q)$. Thus for $|q| = 1$,

$$\begin{aligned}
|f(q)| &= \left| a_0 + \sum_{\ell=1}^{n} q^\ell (a_\ell - a_{\ell-1}) \right| \leq |a_0| + \sum_{\ell=1}^{n} |q|^\ell |a_\ell - a_{\ell-1}| \\
&= |a_0| + \sum_{\ell=1}^{n} |a_\ell - a_{\ell-1}| = |a_0| + M_r + \sum_{\ell=r+1}^{p} |a_\ell - a_{\ell-1}| + M_p \\
&\leq |a_0| + M_r + \sum_{\ell=r+1}^{p} (|\alpha_\ell - \alpha_{\ell-1}| + |\beta_\ell - \beta_{\ell-1}| + |\gamma_\ell - \gamma_{\ell-1}| + |\delta_\ell - \delta_{\ell-1}|) + M_p \\
&= |a_0| + M_r + |\alpha_{r+1} - \rho_{R_1}\alpha_r + \rho_{R_1}\alpha_r - \alpha_r| + \sum_{\ell=r+2}^{\eta-1} |\alpha_\ell - \alpha_{\ell-1}| \\
&\quad + |\alpha_\eta - k_R\alpha_\eta + k_R\alpha_\eta - \alpha_{\eta-1}| + |\alpha_{\eta+1} - k_R\alpha_\eta + k_R\alpha_\eta - \alpha_\eta| \\
&\quad + \sum_{\ell=\eta+2}^{p-1} |\alpha_\ell - \alpha_{\ell-1}| + |\alpha_p - \rho_{R_2}\alpha_p + \rho_{R_2}\alpha_p - \alpha_{p-1}| \\
&\quad + |\beta_{r+1} - \rho_{I_1}\beta_r + \rho_{I_1}\beta_r - \beta_r| + \sum_{\ell=r+2}^{\eta-1} |\beta_\ell - \beta_{\ell-1}| \\
&\quad + |\beta_\eta - k_I\beta_\eta + k_I\beta_\eta - \beta_{\eta-1}| + |\beta_{\eta+1} - k_I\beta_\eta + k_I\beta_\eta - \beta_\eta| \\
&\quad + \sum_{\ell=\eta+2}^{p-1} |\beta_\ell - \beta_{\ell-1}| + |\beta_p - \rho_{I_2}\beta_p + \rho_{I_2}\beta_p - \beta_{p-1}| \\
&\quad + |\gamma_{r+1} - \rho_{J_1}\gamma_r + \rho_{J_1}\gamma_r - \gamma_r| + \sum_{\ell=r+2}^{\eta-1} |\gamma_\ell - \gamma_{\ell-1}| \\
&\quad + |\gamma_\eta - k_J\gamma_\eta + k_J\gamma_\eta - \gamma_{\eta-1}| + |\gamma_{\eta+1} - k_J\gamma_\eta + k_J\gamma_\eta - \gamma_\eta| \\
&\quad + \sum_{\ell=\eta+2}^{p-1} |\gamma_\ell - \gamma_{\ell-1}| + |\gamma_p - \rho_{J_2}\gamma_p + \rho_{J_2}\gamma_p - \gamma_{p-1}| \\
&\quad + |\delta_{r+1} - \rho_{K_1}\delta_r + \rho_{K_1}\delta_r - \delta_r| + \sum_{\ell=r+2}^{\eta-1} |\delta_\ell - \delta_{\ell-1}| \\
&\quad + |\delta_\eta - k_K\delta_\eta + k_K\delta_\eta - \delta_{\eta-1}| + |\delta_{\eta+1} - k_K\delta_\eta + k_K\delta_\eta - \delta_\eta| \\
&\quad + \sum_{\ell=\eta+2}^{p-1} |\delta_\ell - \delta_{\ell-1}| + |\delta_p - \rho_{K_2}\delta_p + \rho_{K_2}\delta_p - \delta_{p-1}| + M_p \\
&\leq |a_0| + M_r + |\alpha_{r+1} - \rho_{R_1}\alpha_r| + |\rho_{R_1}\alpha_r - \alpha_r| - \alpha_{r+1} + \alpha_{\eta-1} + |\alpha_\eta - k_R\alpha_\eta|
\end{aligned}$$

$$
\begin{aligned}
&+|k_R\alpha_\eta-\alpha_{\eta-1}|+|\alpha_{\eta+1}-k_R\alpha_\eta|+|k_R\alpha_\eta-\alpha_\eta|+\alpha_{\eta+1}-\alpha_{p-1}+|\alpha_p-\rho_{R_2}\alpha_p|\\
&+|\rho_{R_2}\alpha_p-\alpha_{p-1}|+|\beta_{r+1}-\rho_{I_1}\beta_r|+|\rho_{I_1}\beta_r-\beta_r|-\beta_{r+1}+\beta_{\eta-1}+|\beta_\eta-k_I\beta_\eta|\\
&+|k_I\beta_\eta-\beta_{\eta-1}|+|\beta_{\eta+1}-k_I\beta_\eta|+|k_I\beta_\eta-\beta_\eta|+\beta_{\eta+1}-\beta_{p-1}+|\beta_p-\rho_{I_2}\beta_p|\\
&+|\rho_{I_2}\beta_p-\beta_{p-1}|+|\gamma_{r+1}-\rho_{J_1}\gamma_r|+|\rho_{J_1}\gamma_r-\gamma_r|-\gamma_{r+1}+\gamma_{\eta-1}+|\gamma_\eta-k_J\gamma_\eta|\\
&+|k_J\gamma_\eta-\gamma_{\eta-1}|+|\gamma_{\eta+1}-k_J\gamma_\eta|+|k_J\gamma_\eta-\gamma_\eta|+\gamma_{\eta+1}-\gamma_{p-1}\\
&+|\gamma_p-\rho_{J_2}\gamma_p|+|\rho_{J_2}\gamma_p-\gamma_{p-1}|+|\delta_{r+1}-\rho_{K_1}\delta_r|+|\rho_{K_1}\delta_r-\delta_r|-\delta_{r+1}+\delta_{\eta-1}\\
&+|\delta_\eta-k_K\delta_\eta|+|k_K\delta_\eta-\delta_{\eta-1}|+|\delta_{\eta+1}-k_K\delta_\eta|+|k_K\delta_\eta-\delta_\eta|\\
&+\delta_{\eta+1}-\delta_{p-1}+|\delta_p-\rho_{K_2}\delta_p|+|\rho_{K_2}\delta_p-\delta_{p-1}|+M_p\\
=\;&|a_0|+M_r-\rho_{R_1}\alpha_r+|\alpha_r|(1-\rho_{R_1})+2|\alpha_\eta|(k_R-1)+2k_R\alpha_\eta+|a_p|(1-\rho_{R_2})\\
&-\rho_{R_2}\alpha_p-\rho_{I_1}\beta_r+|\beta_r|(1-\rho_{I_1})+2|\beta_\eta|(k_I-1)+2k_I\beta_\eta+|a_p|(1-\rho_{I_2})-\rho_{I_2}\beta_p\\
&-\rho_{J_1}\gamma_r+|\gamma_r|(1-\rho_{J_1})+2|\gamma_\eta|(k_J-1)+2k_J\gamma_\eta+|a_p|(1-\rho_{J_2})-\rho_{J_2}\gamma_p\\
&-\rho_{K_1}\delta_r+|\delta_r|(1-\rho_{K_1})+2|\delta_\eta|(k_K-1)+2k_K\delta_\eta+|a_p|(1-\rho_{K_2})-\rho_{K_2}\delta_p+M_p\\
=\;&M.
\end{aligned}
$$

Notice that $q^n f(1/q)$ has the same bound on $|q|=1$ as $f(q)$. So, by Theorem 3.2, for $|q|\le 1$ we have $|q^n f(1/q)|\le M$ and hence $|f(1/q)|\le M/|q|^n$. Replacing q with $1/q$ we have $|f(q)|\le M|q|^n$ for $|q|\ge 1$. Hence for $|q|\ge 1$,

$$|P(q)*(1-q)|=|f(q)-q^{n+1}a_n|\ge |q^{n+1}||a_n|-|f(q)|\ge |q^{n+1}||a_n|-M|q|^n=|q|^n(|q||a_n|-M).$$

So if $|q|>M/|a_n|$ then $P(q)*(1-q)\neq 0$. Therefore, all zeros of $P(q)$ lie in $|q|\le M/|a_n|$, as claimed.

Next, consider $S(q)=q^n*P(1/q)=\sum_{\ell=0}^{n} q^{n-\ell}a_\ell$ and let

$$H(q)=S(q)*(1-q)=-a_0q^{n+1}+\sum_{\ell=1}^{n} q^{n+1-\ell}(a_{\ell-1}-a_\ell)+a_n.$$

Then

$$
\begin{aligned}
|H(q)|\;&\ge\; |q|^{n+1}|a_0|-\left\{\sum_{\ell=1}^{n}|q|^{n+1-\ell}|a_{\ell-1}-a_\ell|+|a_n|\right\}\\
&=\; |q|^{n+1}|a_0|-\left\{\sum_{\ell=1}^{r}|q|^{n+1-\ell}|a_{\ell-1}-a_\ell|+\sum_{\ell=r+1}^{p}\Big(|q|^{n+1-\ell}|\alpha_{\ell-1}-\alpha_\ell|\right.
\end{aligned}
$$

$$
\begin{aligned}
& +|q|^{n+1-\ell}|\beta_{\ell-1}-\beta_\ell| + |q|^{n+1-\ell}|\gamma_{\ell-1}-\gamma_\ell| + |q|^{n+1-\ell}|\delta_{\ell-1}-\delta_\ell|\Big) \\
& + \sum_{\ell=p+1}^{n} |q|^{n+1-\ell}|a_{\ell-1}-a_\ell| + |a_n|\Bigg\} \\
\geq\ & |q|^{n+1}|a_0| - \Bigg\{ \sum_{\ell=1}^{r} |q|^{n+1-\ell}|a_{\ell-1}-a_\ell| + |q|^{n-r}|\alpha_r|(1-\rho_{R_1}) \\
& +|q|^{n-r}(\alpha_{r+1}-\rho_{R_1}\alpha_r) + |q|^{n-r}|\beta_r|(1-\rho_{I_1}) + |q|^{n-r}(\beta_{r+1}-\rho_{I_1}\beta_r) \\
& +|q|^{n-r}|\gamma_r|(1-\rho_{J_1}) + |q|^{n-r}(\gamma_{r+1}-\rho_{J_1}\gamma_r) + |q|^{n-r}|\delta_r|(1-\rho_{K_1}) \\
& +|q|^{n-r}(\delta_{r+1}-\rho_{K_1}\delta_r) + \sum_{\ell=r+2}^{\eta-1} \Big(|q|^{n+1-\ell}|\alpha_{\ell-1}-\alpha_\ell| + |q|^{n+1-\ell}|\beta_{\ell-1}-\beta_\ell| \\
& |q|^{n+1-\ell}|\gamma_{\ell-1}-\gamma_\ell| + |q|^{n+1-\ell}|\delta_{\ell-1}-\delta_\ell|\Big) + |q|^{n+1-\eta}(k_R\alpha_\eta-\alpha_{\eta-1}) \\
& +|q|^{n+1-\eta}|\alpha_\eta|(k_R-1) + |q|^{n+1-\eta}(k_I\beta_\eta-\beta_{\eta-1}) + |q|^{n+1-\eta}|\beta_\eta|(k_I-1) \\
& +|q|^{n+1-\eta}(k_J\gamma_\eta-\gamma_{\eta-1}) + |q|^{n+1-\eta}|\gamma_\eta|(k_J-1) + |q|^{n+1-\eta}(k_K\delta_\eta-\delta_{\eta-1}) \\
& +|q|^{n+1-\eta}|\delta_\eta|(k_K-1) + |q|^{n-\eta}|\alpha_\eta|(k_R-1) + |q|^{n-\eta}(k_R\alpha_\eta-\alpha_{\eta+1}) \\
& +|q|^{n-\eta}|\beta_\eta|(k_I-1) + |q|^{n-\eta}(k_I\beta_\eta-\beta_{\eta+1}) + |q|^{n-\eta}|\gamma_\eta|(k_J-1) \\
& +|q|^{n-\eta}(k_J\gamma_\eta-\gamma_{\eta+1}) + |q|^{n-\eta}|\delta_\eta|(k_K-1) + |q|^{n-\eta}(k_K\delta_\eta-\delta_{\eta+1}) \\
& + \sum_{\ell=\eta+2}^{p-1} \Big(|q|^{n+1-\ell}|\alpha_{\ell-1}-\alpha_\ell| + |q|^{n+1-\ell}|\beta_{\ell-1}-\beta_\ell| + |q|^{n+1-\ell}|\gamma_{\ell-1}-\gamma_\ell| \\
& +|q|^{n+1-\ell}|\delta_{\ell-1}-\delta_\ell|\Big) + |q|^{n+1-p}(\alpha_{p-1}-\rho_{R_2}\alpha_p) + |q|^{n+1-p}|\alpha_p|(1-\rho_{R_2}) \\
& +|q|^{n+1-p}(\beta_{p-1}-\rho_{I_2}\beta_p) + |q|^{n+1-p}|\beta_p|(1-\rho_{I_2}) + |q|^{n+1-p}(\gamma_{p-1}-\rho_{J_2}\gamma_p) \\
& +|q|^{n+1-p}|\gamma_p|(1-\rho_{J_2}) + |q|^{n+1-p}(\delta_{p-1}-\rho_{K_2}\delta_p) + |q|^{n+1-p}|\delta_p|(1-\rho_{K_2}) \\
& + \sum_{\ell=p+1}^{n} |q|^{n+1-\ell}|a_{\ell-1}-a_\ell| + |a_n|\Bigg\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|H(q)| \geq\ & |q|^n\Bigg[|q||a_0| - \Bigg\{ \sum_{\ell=1}^{r} |q|^{1-\ell}|a_{\ell-1}-a_\ell| + |q|^{-r}|\alpha_r|(1-\rho_{R_1}) \\
& +|q|^{-r}(\alpha_{r+1}-\rho_{R_1}\alpha_r) + |q|^{-r}|\beta_r|(1-\rho_{I_1}) + |q|^{-r}(\beta_{r+1}-\rho_{I_1}\beta_r) \\
& +|q|^{-r}|\gamma_r|(1-\rho_{J_1}) + |q|^{-r}(\gamma_{r+1}-\rho_{J_1}\gamma_r) + |q|^{-r}|\delta_r|(1-\rho_{K_1}) \\
& +|q|^{-r}(\delta_{r+1}-\rho_{K_1}\delta_r) + \sum_{\ell=r+2}^{\eta-1} \Big(|q|^{1-\ell}|\alpha_{\ell-1}-\alpha_\ell| + |q|^{1-\ell}|\beta_{\ell-1}-\beta_\ell| \\
& +|q|^{1-\ell}|\gamma_{\ell-1}-\gamma_\ell| + |q|^{1-\ell}|\delta_{\ell-1}-\delta_\ell|\Big) + |q|^{1-\eta}(k_R\alpha_\eta-\alpha_{\eta-1})
\end{aligned}
$$

$$
\begin{aligned}
&+|q|^{1-\eta}|\alpha_\eta|(k_R-1)+|q|^{1-\eta}(k_I\beta_\eta-\beta_{\eta-1})+|q|^{1-\eta}|\beta_\eta|(k_I-1)\\
&+|q|^{1-\eta}(k_J\gamma_\eta-\gamma_{\eta-1})+|q|^{1-\eta}|\gamma_\eta|(k_J-1)+|q|^{1-\eta}(k_K\delta_\eta-\delta_{\eta-1})\\
&+|q|^{1-\eta}|\delta_\eta|(k_K-1)+|q|^{-\eta}|\alpha_\eta|(k_R-1)+|q|^{-\eta}(k_R\alpha_\eta-\alpha_{\eta+1})\\
&+|q|^{-\eta}|\beta_\eta|(k_I-1)+|q|^{-\eta}(k_I\beta_\eta-\beta_{\eta+1})+|q|^{-\eta}|\gamma_\eta|(k_J-1)\\
&+|q|^{-\eta}(k_J\gamma_\eta-\gamma_{\eta+1})+|q|^{-\eta}|\delta_\eta|(k_K-1)+|q|^{-\eta}(k_K\delta_\eta-\delta_{\eta+1})\\
&+\sum_{\ell=\eta+2}^{p-1}\big(|q|^{1-\ell}|\alpha_{\ell-1}-\alpha_\ell|+|q|^{1-\ell}|\beta_{\ell-1}-\beta_\ell|+|q|^{1-\ell}|\gamma_{\ell-1}-\gamma_\ell|\\
&+|q|^{1-\ell}|\delta_{\ell-1}-\delta_\ell|\big)+|q|^{1-p}(\alpha_{p-1}-\rho_{R_2}\alpha_p)+|q|^{1-p}|\alpha_p|(1-\rho_{R_2})\\
&+|q|^{1-p}(\beta_{p-1}-\rho_{I_2}\beta_p)+|q|^{1-p}|\beta_p|(1-\rho_{I_2})+|q|^{1-p}(\gamma_{p-1}-\rho_{J_2}\gamma_p)\\
&+|q|^{1-p}|\gamma_p|(1-\rho_{J_2})+|q|^{1-p}(\delta_{p-1}-\rho_{K_2}\delta_p)+|q|^{1-p}|\delta_p|(1-\rho_{K_2})\\
&+\sum_{\ell=p+1}^{n}|q|^{1-\ell}|a_{\ell-1}-a_\ell|+|a_n|/|q|^n\Bigg\}\Bigg].
\end{aligned}
$$

For $|q|>1$, and hence $1/(|q|^{n-\ell})\leq 1$ for $0\leq\ell<n$, we have

$$
\begin{aligned}
|H(q)| \;\geq\; & |q|^n\Bigg[|q||a_0|-\Bigg\{M_r+|\alpha_r|(1-\rho_{R_1})-\rho_{R_1}\alpha_r+|\beta_r|(1-\rho_{I_1})-\rho_{I_2}\beta_r\\
&+|\gamma_r|(1-\rho_{J_1})-\rho_{J_1}\gamma_r+|\delta_r|(1-\rho_{K_1})-\rho_{K_1}\delta_r+2k_R\alpha_\eta+2|\alpha_\eta|(k_R-1)\\
&+2k_I\beta_\eta+2|\beta_\eta|(k_I-1)+2k_J\gamma_\eta+2|\gamma_\eta|(k_J-1)+2k_K\delta_\eta+2|\delta_\eta|(k_K-1)\\
&-\rho_{R_2}\alpha_p+|\alpha_p|(1-\rho_{R_2})-\rho_{I_2}\beta_p+|\beta_p|(1-\rho_{R_2})-\rho_{J_2}\gamma_p+|\gamma_p|(1-\rho_{J_2})\\
&-\rho_{K_2}\delta_p+|\delta_p|(1-\rho_{K_2})+M_p+|a_n|\Bigg\}\Bigg]=|q|^n(|q||a_0|-(M-|a_0|+|a_n|)).
\end{aligned}
$$

Notice that $|H(q)|\geq|q|^n(|q||a_0|-(M-|a_0|+|a_n|))>0$ if $|q|>(M-|a_0|+|a_n|)/|a_0|$. Thus all zeros of $H(q)$ whose modulus is greater than 1 lie in $|q|\leq(M-|a_0|+|a_n|)/|a_0|$. So all zeros of $H(q)$ and hence of $S(q)$ lie in $|q|\leq\max\{1,(M-|a_0|+|a_n|)/|a_0|\}$. Therefore all zeros of $P(q)$ lie in $|q|\geq\min\{1,|a_0|/(M-|a_0|+|a_n|)$, as claimed. □

References

[1] A. Aziz and B. A. Zargar, Bounds for the zeros of a polynomial with restricted coefficients," *Applied Mathematics*, **3** (2012), 30–33.

[2] N. Carney, R. Gardner, R. Keaton, A. Powers, The Eneström-Kakeya theorem for polynomials of a quaternionic variable, *Journal of Approximation Theory*, **250** (2020), Article 105325, 10 pp.

[3] T. Chan and M. Malik, On Erdős-Lax Theorem, *Proceedings of the Indian Academy of Sciences*, **92** (1983), 191–193.

[4] G. Eneström, Härledning af en allmän formel för antalet pensionärer, som vid en godtyeklig tidpunkt förefinnas inom en sluten pensionslcassa, *Övfers. Vetensk.-Akad. Fórhh.*, **50** (1893), 405–415.

[5] R. Gardner and M. Gladin, Generalizations of the Enestrom-Kakeya Theorem Involving Weakened Hypotheses, *AppliedMath*, **2**(4) (2022), 687–699.

[6] R. Gardner and N. K. Govil, On the Location of the Zeros of a Polynomial, *Journal of Approximation Theory*, **78** (1994), 286–292.

[7] G. Gentili and D. Struppa, A new theory of regular functions of a quaternionic variable, *Advances in Mathematics*, **216** (2007), 279–301.

[8] S. Kakeya, On the limits of the roots of an algebraic equation with positive coefficients, *Tôhoku Math. J. First Ser.*, **2** (1912–1913), 140–142.

[9] T. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics, vol. 123, Springer-Verlag, 1991.

[10] M. Qazi, On the maximum modulus of polynomials, *Proceedings of the American Mathematical Society*, **115** (1992), 337–343.

[11] M. A. Shah, R. Swroop, H. M. Sofi, I. Nisar, A generalization of Eneström-Kakeya Theorem and a zero free region of a polynomial, *Journal of Applied Mathematics and Physics*, **9** (2021), 1271–1277.

[12] D. Tripathi, A note on Eneström-Kakeya theorem for a polynomial with quaternionic variable, *Arabian Journal of Mathematics*, **9** (2020), 707–714.

Robert Gardner (gardnerr@etsu.edu) and Matthew Gladin (gladin@etsu.edu)

Department of Mathematics and Statistics

East Tennessee State University

Johnson City, Tennessee $37614 - 0663$