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# Commentary by Robert Gardner

Dr. Rawling states two versions of the two envelopes problem. In both versions, an amount of money is placed in an envelope $O$, a coin is tossed and twice the amount of money in $O$ is placed in envelope $T$ if the coin comes up heads or half the amount of money in $O$ is placed in envelope $O$ is placed in envelope $T$ if the coin comes up tails. As he observes, if the amount of money in $O$ is \$10 (or any known quantity $x$), then the expected amount in $T$ is $\$1.25 \times 10$ (or $\$1.25x$ in general). Therefore, in such a situation no "exchange paradox" arises. However, if the amount of money in $O$ is not known (say it is some value $2^n$ where $n > 0$ is an integer), then he argues that this leads to the exchange paradox in which if one holds envelope $O$, then s/he deduces that the expected amount in envelope $T$ is greater, and conversely if one holds envelope $T$, then s/he deduces that the expected amount in evelope $O$ is greater. Since this is clearly contradictory, Dr. Rawling is lead to the conclusion that he must either reject

1. P(tails | $T$ contains $\$2^n$) = P(heads | $T$ contains $\$2^n$) = 0.5, or

2. countable additivity.

I propose that a third option exists to resolve the paradox which is perhaps even more elementary.

If we interprete the two envelopes problem to consist of the following events (in order):

1. put an amount of money $x$ in envelope $O$,

2. flip a coin,

3. put twice the amount $x$ in envelope $T$ if the coin comes up heads, and put half the amount $x$ in envelope $T$ if the coin comes up tails,

then the exchange paradox is easily explained. The real problem lies in determining *how* the amount $x$ which is to be placed in envelope $O$ is to be determined. If we are given a probability distribution that describes how the quantity is chosen, then the paradox immediately dissappears. Suppose this probability distribution has mean $\$\mu$. Then the expected amount in envelope $O$ is $\$\mu$ and the expected amount in $T$ is $\$\frac{5}{4}\mu$. No paradox exists and one should choose envelope $T$. Notice this can be accomplished over a countable sample space for which the probability of no event is zero and yet we still have countable additivity and finite expected values, as illustrated in the following example.

**Example 1.** Suppose a positive integer $n$ is chosen according to the probability distribution $p(n) = \frac{1}{2^n}$ and $\$\left(\frac{3}{2}\right)^n$ is placed in envelope $O$. A coin is tossed and an amount placed in envelope $T$ as described above. Then the expected amount in $O$ is $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n\left(\frac{1}{2^n}\right) = 3$ and the expected amount in $T$ is $1.25 \times 3 = \frac{15}{4}$. It is fallacious to argue that the expected value of $O$ is $\frac{5}{4}$ times the expected value of $O$, since the expected value of $O$ is already determined by the probability distribution.

Of course, one can argue that the game given in Example 1 cannot actually be played since it requires the availability of an unbounded amount of money. In the following example, this problem does not arise.

**Example 2.** Suppose a positive number between 0 and 1 is chosen according to a uniform distribution (i.e. the probability function if $f(x) = 1$) is placed in envelope $O$. Then the probability that a value from set $A \subset (0,1)$ is chosen is $\int_A 1$ (for generality, we take the integration to be Lebesgue integration). The expected amount in $O$ is $\int_{(0,1)} x = \frac{1}{2}$. Following the coin toss as described above, the expected amount in envelope $T$ is $\frac{5}{8}$. Again, there is no paradox and one should choose envelope $T$. Notice that in this example, we have countable additivity (since Lebesgue integration is countably additive), although the sample space is uncountable.

Notice that one can argue that $eav(T) = \frac{5}{4}eav(O)$. However, as above, it is not valid to calculate $eav(O)$ in terms of $eav(T)$ as $eav(O) = \frac{e}{4}eav(T)$, since $eav(O)$ is given by the probability distribution. In addition, we still have $p(\text{tails} \mid T \text{ contains } x) = p(\text{heads} \mid T \text{ contains } x) = 0.5$.

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# Rawling's Reply to Gardner

I thank Professor Gardner for his commentary. I shall here respond to him on two issues, largely for purposes of clarification.

Professor Gardner suggests that

$$\text{(A)} \qquad p(\text{tails} \mid T \text{ contains } \$2n) = p(\text{heads} \mid T \text{ contains } \$2n) = 0.5$$

is consistent with countable additivity. I demur, given the proviso (which is an initial condition of the problem):

$$\text{(B)} \qquad p(\text{tails} \mid O \text{ contains } \$2n) = p(\text{heads} \mid O \text{ contains } \$2n) = 0.5.$$