Horizontal asymptotes: What they are not

A common misconception about a horizontal asymptote of a function is that the function "gets closer and closer to the asymptote but never gets there." Expressed in the appropriate "mathematese" we have the following:

FAUX DEFINITION. Let $f(x)$ be a function defined for all $x \in (-\infty, \infty)$. Then, the line $y = k$ is a horizontal asymptote of f if, as x increases, the graph of f gets closer and closer to $y = k$ but $f(x) \neq k$ for all $x \in (-\infty, \infty)$.

We can easily represent several flaws in this "definition." We do so in the following two examples.

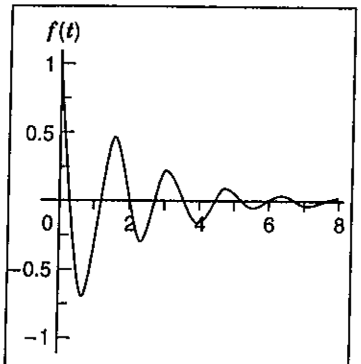

Fig. 1
A graph of $f(t) = e^{-t/2} \cos 4t$

Example 1. Suppose that a weight is attached to a vertically mounted spring and that the weight is set in motion by displacing it vertically from its equilibrium position and then releasing it. If we let $f(t)$ be the displacement of the weight from its equilibrium position at time t, then, assuming the presence of friction and with the appropriate choice of constants and units, we find that $f(t) = e^{-t/2} \cos 4t$. (See **fig. 1.**) Notice that f has a horizontal asymptote of $y = 0$. Also, f takes on the value 0 for all values of t such that

$$t = \frac{2n+1}{8}\pi$$

for some integer n. Certainly this example illustrates that f can "get there"! This example also illustrates the falseness of the "gets closer and closer" assumption.

Example 1, which illustrates the so-called *damped harmonic motion,* is especially appealing because of its physical origin. Students can easily visualize masses oscillating on springs. A less physically interesting, but equally valid, example is the following.

Example 2. Consider $g(x) = e^{-x}$ and the line $y = -1$. (See **fig. 2.**) Since g is a decreasing positive function, it does in fact get closer and closer to the line $y = -1$ as x increases. Also, g never attains the value -1. However, $y = -1$ is not a horizontal asymptote of g.

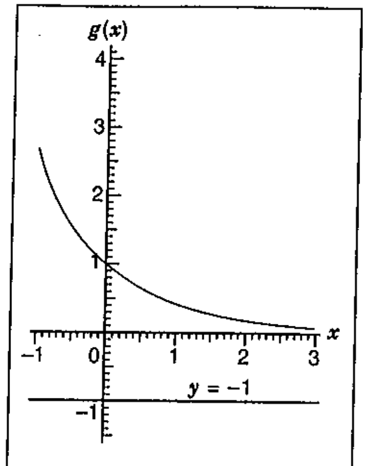

Fig. 2
A graph of $g(x) = e^{-x}$

Example 2 illustrates that an accurate definition of horizontal asymptote must somehow replace the "gets closer and closer" concept with a "gets arbitrarily close" concept.

From these examples, we see that a soft approach to the definition of horizontal asymptote can be problematic. Only with the rigor of limits can this difficult concept be dealt with precisely.

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