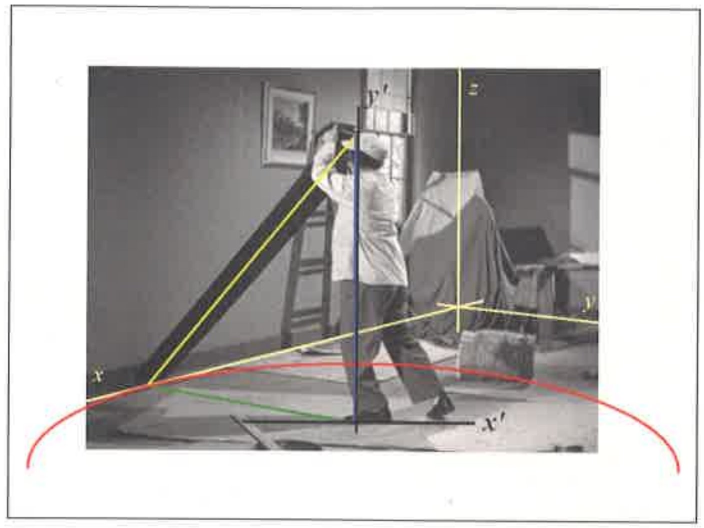
MATHEMATICAL lens

Robert Gardner and Robert Davidson

The Three Stooges Meet the Conic Sections

Photograph 1
Larry props up a board against a wall in the Three Stooges film *A Bird in the Head* (at 3 min. 40 sec.).



Photograph 2
When viewed in perspective, Larry and the board determine a right triangle.

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit http://www.nctm.org/publications/content.aspx?id=10440#lens.

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The Three Stooges comedy team was involved in show business from the 1920s (as a vaudeville act) through the early 1970s. The 190 short films that they recorded between 1934 and 1958 (films still aired on various cable television channels) made them iconic figures.

Their 1946 film *A Bird in the Head* is representative of the style of Stooges' humor. In the opening scene, Moe, Larry, and Curly are wallpapering a room. Moe trips over a board lying on the floor. He tells Larry to prop up the board. Larry leans the board vertically against the wall, where it stays, precariously, for a few seconds. The board then tips over and crashes to the floor. Moe yells at Larry, Larry props up the board against the wall again, and again it falls. Moe props up the board again (reflecting the Stooges' profound misunderstanding of physics), calls Larry over, and slaps him in the face. The board falls a third time. This time as it falls, it hits Moe on the head—a rare moment of Stooge justice. The scene ends with the room covered in wallpaper (including windows, doors, and bookshelves) and an angry client.

Photograph 1 (3 min. 40 sec. into the film) shows Larry propping up the board against the wall. The lower end of the board is on the floor and against the wall. The upper end is at the height of Larry's head. Therefore, the board, Larry's body, and a line segment from the lower end of the board to Larry's feet form a right triangle in the room. However, this triangle does not appear to be a right triangle in the plane of the photograph. The purpose of this article is to pose a sequence of questions that lead to an approximation of the length of the board. Answering the questions requires knowledge of conic sections, geometry, implicit differentiation, the quadratic equation, and scaling.

1. (*a*) Use the walls of the room to introduce a right-handed x, y, z-coordinate system. Place the x-axis along the line where the floor and the wall facing Larry meet. Trace the axes onto **photograph 1**. Thus, the floor of the room is the xy-plane, the wall that Larry is facing is the xz-plane, and the back wall is the yz-plane.

   (*b*) Draw a line segment along the center of the board from its base to its top (the top of the board roughly coincides with the top of Larry's head). Draw a line segment from the top of Larry's head straight down to the floor (determined by the location of Larry's left foot). Draw a line segment along the floor from Larry's foot to the base of the board. These three line segments determine a right triangle in the x, y, z-coordinate system.

2. We want to find the length of the hypotenuse of the right triangle of part 1. To do so, we pretend that Larry rotates 360° while pivoting on his left foot and while holding the board so that it makes a constant angle with the floor. In this way, the bottom of the board will trace out a circle on the room's floor (i.e., in the xy-plane). Since a circle viewed in perspective is an ellipse, the circle determines an ellipse in the plane of view of the camera.

   (*a*) Introduce a new set of axes in the plane of the photograph: an x'-axis and a y'-axis, in which the origin of this system is at Larry's left foot, the x'-axis is horizontal, and the y'-axis is vertical.

   (*b*) Measure the length of the line segment from Larry's left foot to the top of his head. In the solution presented below, we give such measurements in terms of "units," in which the width of the photograph is 100 such units. Doing so allows us to present this project using any size reproduction of **photograph 1** (although the aspect ratio must be preserved).

   (*c*) From *Stooges among Us* (Davis and Davis 2008, p. 10), we learn that "like Moe, Larry was only five-feet-four inches tall." In other words, Larry is about 5.33 feet tall. This information provides us with the scale of the photograph for any object in the room lying in the $x'y'$-plane. We will denote the scaling factor as s and measure it in units/ft. Use Larry's height from part 2(*c*) to calculate the scaling factor *s*.

(*d*) For reference, sketch onto the photograph an ellipse that corresponds to the circle mentioned earlier. This ellipse is tangent to the *x*-axis at the point at the bottom of the board. Denote this point of tangency in the $x'y'$-coordinate system as (x'_0, y'_0). Measure the photograph to find the coordinates of point (x'_0, y'_0).

3. The center of the ellipse does not correspond to the center of the circle because the front part of the circle is closer to the camera than is the back part of the circle. We denote the radius of the circle as *r*. The plane containing Larry and the camera lens is given in **figure 1**. Let *d* denote the horizontal distance from Larry's left foot to the camera, and let *h* denote the vertical distance labeled in **figure 1**. Use the location of the camera to project the radius of the circle from Larry's left foot to the point on the circle farthest from the camera onto the vertical line through Larry's left foot; denote the resulting distance as y_1'. Project the radius of the circle from Larry's left foot to the point on the circle closest to the camera onto the vertical line through Larry's left foot; denote the resulting distance as y_2'.

We will now set up relationships between y_1', y_2', and other parameters of the ellipse. The minor axis of the ellipse has length $y_1' + y'_2$, and the center of the ellipse in the $x'y'$-coordinate system is $(x', y') = (0, (y_1' - y_2')/2)$. Therefore, the equation of the ellipse is

$$\frac{(x')^2}{a^2} + \frac{\left(y' - \frac{y_1' - y_2'}{2}\right)^2}{b^2} = 1$$

or

$$A(x')^2 + B\left(y' - \frac{y_1' - y_2'}{2}\right)^2 = 1$$

where $A = 1/a^2$ and $B = 1/b^2$. We will measure distances in the studio (such as *r*, *h*, and *d*) in feet. We need the

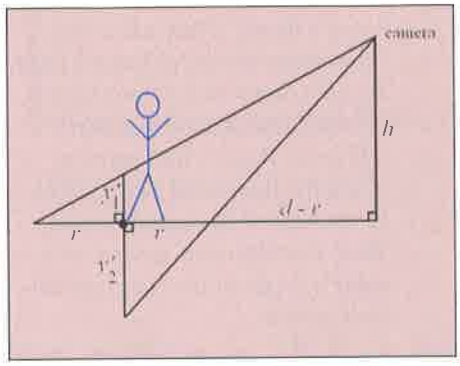

Fig. 1 The plane contains Larry (represented by the stick figure) and the camera.

scaling factor *s* to convert *r*, *h*, and *d* into the units used in the $x'y'$-plane. In this plane, the point $(rs, 0)$ lies on the ellipse. Therefore, the ellipse looks as it does in **figure 2**. We now set up several equations using the symbols we have introduced, instead of the measured numerical values of the symbols. In this way, we avoid rounding errors and ugly numerical manipulations. We will substitute values near the end of our computations.

(*a*) Substitute the coordinates of the point $(rs, 0)$ into the second equation of the ellipse to create a new equation labeled (1). Substitute the coordinates of the point (x'_0, y'_0) into the equation of the ellipse to create another equation; label it (2).

(*b*) Denote the slope of the *x*-axis in the $x'y'$-plane as *m*. Determine *m* from the photograph. Use implicit differentiation and the fact that at the point (x'_0, y'_0) we have

$$\left.\frac{dy'}{dx'}\right|_{(x'_0, y'_0)} = m$$

to set up an equation; denote it as (3).

(*c*) Solve equation (3) for *A*. Substitute this expression for *A* into equation (1) and solve for *B*. Substitute these values of *A* and *B* into equation (2) and call the resulting equation (4).

(*d*) Use similar triangles to express y_1' and y_2' in terms of *d*, *r*, and *h*. Use the scaling factor *s* of part 2(*c*) to

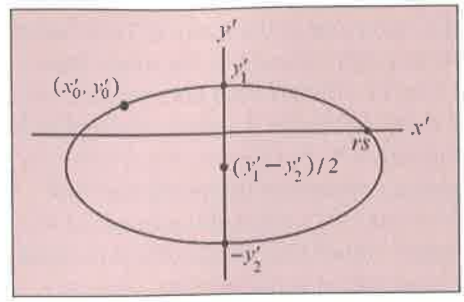

Fig. 2 Viewed in the $x'y'$-plane, an ellipse is traced by the end of the board as Larry rotates on his left foot.

express y_1' and y_2' in the units of the $x'y'$-plane.

4. We now create a polynomial equation that, when solved, will enable us to determine the length of the board.

(*a*) Substitute the scaled values of y_1' and y_2' into equation (4). Simplify this new equation to produce a polynomial equation in $R = r^2$. Notice that the unit of *R* is ft.^{2} and the unit of *r* is ft.

(*b*) Consider the line determined by the bottom of the picture hanging on the wall that Larry faces. Extend this line and the *x*-axis to find where they intersect (this point of intersection will lie off the page containing the photograph). This is a "point at infinity" and lies along the horizon of the photograph. The y'-coordinate of this point at infinity corresponds to the height of the camera, which, in terms of the units of the $x'y'$-plane, is *hs*. Find *hs* and *h*. We cannot determine the horizontal distance from the camera to Larry, *d*, so we assume that $d = 15$ ft.

(*c*) Substitute numerical values for the symbols we have introduced and solve the resulting polynomial equation of part 4(*a*). If necessary, use a computer algebra system to solve the equation. However, the equation can be solved using the quadratic formula.

(*d*) Use the value of $R = r^2$ and Larry's height to calculate the length of the board.

MATHEMATICAL LENS solutions

1. (*a*) See **photograph 2**.
   (*b*) See **photograph 2**.

2. (*a*) See **photograph 2**.

   (*b*) For all measurement in these solutions, we assume that the width of **photograph 1** is 100 units, and we express all measurements in terms of these units. Using this convention, we see that the length of the line segment corresponding to Larry is 55.0 units.

   (*c*) Larry is about 5.33 feet tall, and this height corresponds to 55.0 units, so the scaling factor is

$$s = \frac{55.0 \text{ units}}{5.33 \text{ ft.}} \approx 10.32 \text{ units/ft.}$$

   (*d*) See **photograph 2**. In terms of the units introduced, the coordinates of the point (x'_0, y'_0) are (–38.3, 8.7).

3. (*a*) From point $(rs, 0)$, we have

$$Ar^2s^2 + B\left(0 - \frac{y'_1 - y'_2}{2}\right)^2 = Ar^2s^2 + B\left(\frac{y'_1 - y'_2}{2}\right)^2 = 1. \qquad (1)$$

   From point (x'_0, y'_0), we have

$$A\left(x'_0\right)^2 + B\left(y'_0 - \frac{y'_1 - y'_2}{2}\right)^2 = 1. \qquad (2)$$

   (*b*) By measuring, we find that the point in the plane of the photograph where the x-axis appears to intersect the y'-axis is (0, 17.0). Use this point and point (x'_0, y'_0) = (–38.3, 8.7) to calculate the slope of the x-axis as

$$m = \frac{8.7 - 17.0}{-38.3 - 0} \approx 0.22.$$

   Differentiating the equation for the ellipse implicitly with respect to x' gives

$$2Ax' + 2B\left(y' - \frac{y'_1 - y'_2}{2}\right)\frac{dy'}{dx'} = 0.$$

   At point (x'_0, y'_0) we have

$$2Ax'_0 + 2B\left(y'_0 - \frac{y'_1 - y'_2}{2}\right)m = 0. \qquad (3)$$

(*c*) Now we solve (3) for A:

$$2Ax_0' + 2B\left(y_0' - \frac{y_1' - y_2'}{2}\right)m = 0$$

$$Ax_0' = -B\left(y_0' - \frac{y_1' - y_2'}{2}\right)m$$

$$2Ax_0' = -B\left(2y_0' - y_1' + y_2'\right)m$$

$$A = \frac{Bm}{2x_0'}\left(y_1' - y_2' - 2y_0'\right)$$

We now substitute this expression for A into equation (1) and solve for B:

$$Ar^2s^2 + B\left(\frac{y_1' - y_2'}{2}\right)^2 = 1$$

$$\frac{Bm}{2x_0'}\left(y_1' - y_2' - 2y_0'\right)r^2s^2 + B\left(\frac{y_1' - y_2'}{2}\right)^2 = 1$$

$$B\left[\frac{m\left(y_1' - y_2' - 2y_0'\right)r^2s^2}{2x_0'} + \left(\frac{y_1' - y_2'}{2}\right)^2\right] = 1$$

$$B = \frac{1}{\frac{m(y_1' - y_2' - 2y_0')r^2s^2}{2x_0'} + \left(\frac{y_1' - y_2'}{2}\right)^2}$$

$$B = \frac{4x_0'}{2mr^2s^2(y_1' - y_2' - 2y_0') + x_0'(y_1' - y_2')^2}$$

Therefore, we can express A as

$$A = \frac{2m(y_1' - y_2' - 2y_0')}{2mr^2s^2(y_1' - y_2' - 2y_0') + x_0'(y_1' - y_2')^2}.$$

With these expressions substituted into equation (2), we obtain

$$\begin{aligned} &2m(y_1' - y_2' - 2y_0')(x_0')^2 \\ &\quad + x_0'(y_1' - y_2' - 2y_0')^2 = \\ &2mr^2s^2(y_1' - y_2' - 2y_0') + x_0'(y_1' - y_2')^2 \end{aligned} \qquad (4)$$

(*d*) By similar triangles, we have from **figure 1** that

$$\frac{h}{d+r} = \frac{y_1'}{r} \rightarrow y_1' = \frac{hr}{d+r}$$

and

$$\frac{y_2'}{r} = \frac{h}{d-r} \rightarrow y_2' = \frac{hr}{d-r}.$$

However, these equations give y_1' and y_2' in terms of the units of h, d, and r (namely, feet). We must use the scaling factor s to express y_1' and y_2' in the units of the $x'y'$-plane. This gives

$$y_1' = \frac{hrs}{d+r} \text{ and } y_2' = \frac{hrs}{d-r}.$$

4. (*a*) Substituting y_1' and y_2' into equation (4) gives

$$\begin{aligned} &2m(x_0')^2\left(\frac{2hr^2s}{r^2-d^2} - 2y_0'\right) + \\ &\qquad x_0'\left(\frac{2hr^2s}{r^2-d^2} - 2y_0'\right)^2 = \\ &2mr^2s^2\left(\frac{2hr^2s}{r^2-d^2} - 2y_0'\right) + x_0'\left(\frac{2hr^2s}{r^2-d^2}\right)^2. \end{aligned}$$

After a great deal of painstaking algebraic manipulations and assuming that $r^2 \neq d^2$ (reasonable since **photograph 1** clearly implies that the camera is farther from Larry than the length of the board), we can simplify to

$$\begin{aligned} &R^2\left[ms^2(hs - y_0')\right] \\ &\quad + R\left[\begin{array}{l}(hs - y_0')\left(x_0'y_0' - m(x_0')^2\right) + \\ y_0's\left(md^2s + hx_0'\right)\end{array}\right] \\ &\qquad + \left[x_0'y_0'd^2(y_0' - mx_0')\right] = 0 \end{aligned}$$

where we have set $R = r^2$. The intermediate steps in this derivation can be found at **www.nctm.org/mt010.**

(*b*) We find that $hs = 45.5$ units and therefore that $h = 45.5/s$ ft. = $45.5/10.32$ ft. ≈ 4.41 ft.

(*c*) Using $m = 0.22$, $x'_0 = -38.3$ units, $y'_0 = 8.7$ units, $s = 10.32$ units/ft., $h = 4.41$ ft., and $d = 15.0$ ft., we find from the quadratic formula that $R = r^2 = -41.43$ ft.^{2} or $R = r^2 = 34.98$ ft.^{2}. Therefore, we must have $r^2 = 34.98$ ft.^{2}.

(*d*) We now have the lengths of the two legs of the right triangle determined by Larry and the board. By the Pythagorean theorem, the length of the board is

$$\sqrt{r^2 + 5.33^2} \approx 7.96 \text{ ft.}$$

We conclude that the board is likely the standard length of 8 ft. If we modify the value of d, we find that for $d = 10.0$ ft., we still get (to two decimal places) that the length of the board is 7.96 ft. With $d = 20$ ft., we find that the length of the board changes slightly to 7.80 ft. Therefore, the length of the board is not very sensitive to changes in d, and our estimate of 8 ft. seems accurate.

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For an expanded solution to question 4(*a*), go to the *Mathematics Teacher* home page: **www.nctm.org/mt**.

For a mathematical photograph for which you may create your own questions, go to **www.nctm.org/mt010**. Send your questions to the Mathematical Lens editors.