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Vol. 179, No. 1, October 1993
Printed in Belgium

Inequalities Concerning the L^p Norm of a Polynomial and Its Derivative

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Received March 2, 1992

Let $P_n(z) = a_n \prod_{v=1}^{n} (z - z_v)$, $a_n \neq 0$, be a polynomial of degree n. It has been proved that if $|z_v| \geqslant K_v \geqslant 1$, $1 \leqslant v \leqslant n$, then for $p \geqslant 1$,

$$\left(\int_0^{2\pi} |P_n'(e^{i\theta})|^p \, d\theta\right)^{1/p} \leqslant nF_p \left(\int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta\right)^{1/p},$$

where $F_p = \{2\pi / \int_0^{2\pi} |t_0 + e^{i\theta}|^p \, d\theta\}^{1/p}$, and $t_0 = \{1 + n / \sum_{v=1}^{n} (1/(K_v - 1))\}$. This result generalizes the well known L^p inequality due to De Bruijn for polynomials not vanishing in $|z| < 1$. On making $p \to \infty$, it gives the L^∞ inequality due to Govil and Labelle which as a special case includes the Erdős conjecture proved by Lax.

1. INTRODUCTION AND STATEMENT OF RESULTS

If $P_n(z)$ is a polynomial of degree at most n, then according to a famous result known as Bernstein's inequality (for references, see [6])

$$\max_{|z|=1} |P_n'(z)| \leqslant n \max_{|z|=1} |P_n(z)|. \tag{1.1}$$

Here the equality holds if and only if $P_n(z)$ has all its zeros at the origin. In case $P_n(z)$ does not vanish in $|z| < 1$, it was conjectured by Erdős and proved by Lax [4] that (1.1) can be replaced by

$$\max_{|z|=1} |P_n'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P_n(z)|. \tag{1.2}$$

0022-247X/93 $5.00

The above inequality is sharp with equality holding for polynomials of the form $P_n(z) = \lambda + \mu z^n$, $|\lambda| = |\mu|$.

The L^p analogue of (1.1) was proved by Zygmund [7] and that of (1.2) by De Bruijn [1] who proved that if $P_n(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for $p \geqslant 1$,

$$\left(\int_0^{2\pi} |P_n'(e^{i\theta})|^p \, d\theta\right)^{1/p} \leqslant nC_p \left(\int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta\right)^{1/p}, \tag{1.3}$$

where $C_p = (2\pi/\int_0^{2\pi} |1 + e^{i\theta}|^p \, d\theta)^{1/p}$. This inequality is also sharp and again equality holds for $p(z) = \lambda + \mu z^n$, $|\lambda| = |\mu|$.

Here we consider polynomials having no zeros in $|z| < 1$ and generalize the above result of De Bruijn by obtaining a bound that depends on the location of all the zeros of the polynomial. We prove

THEOREM. *Let $P_n(z) = a_n \prod_{v=1}^n (z - z_v)$, $a_n \neq 0$, be a polynomial of degree n. If $|z_v| \geqslant K_v \geqslant 1$, $1 \leqslant v \leqslant n$, then for $p \geqslant 1$,*

$$\left(\int_0^{2\pi} |P_n'(e^{i\theta})|^p \, d\theta\right)^{1/p} \leqslant nF_p \left(\int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta\right)^{1/p}, \tag{1.4}$$

where $F_p = \{2\pi/\int_0^{2\pi} |t_0 + e^{i\theta}|^p \, d\theta\}^{1/p}$, and $t_0 = \{1 + n/\sum_{v=1}^n (1/(K_v - 1))\}$. The result is best possible in the case $K_v = 1$, $1 \leqslant v \leqslant n$, and the equality holds for the polynomial $p(z) = (z+1)^n$.

If $K_v = 1$ for some v, $1 \leqslant v \leqslant n$, then $t_0 = 1$ and (1.4) reduces to the inequality (1.3) due to De Bruijn [1]. If $K_v \geqslant K \geqslant 1$ for all v, $1 \leqslant v \leqslant n$, then as is easy to verify, $F_p \leqslant \{2\pi/\int_0^{2\pi} |K + e^{i\theta}|^p \, d\theta\}^{1/p}$ and our theorem reduces to Theorem 9 of Govil and Rahman [3].

Remark. The statement of our theorem might suggest that we need to know all the zeros of the polynomial but it is not so. No doubt, the usefulness of the theorem will be heightened if the polynomial is given in terms of its zeros. If in particular we know that the polynomial $P_n(z)$ is the product of two or more polynomials having zeros in $|z| \geqslant K_1 > 1$, $|z| \geqslant K_2 > 1$, etc., the constant t_0 in our theorem would obviously be $> K = \min(K_1, K_2, ...)$ and thus the bound obtained by our theorem will be sharper than the bound obtained from both the known results, De Bruijn's Theorem [1] and Theorem 9 of Govil and Rahman [3].

If in our theorem we make $p \to \infty$, we get the following result due to Govil and Labelle [2], which is a generalization of the Erdös conjecture proved by Lax (see (1.2)).

COROLLARY [2]. *Let $P_n(z)=a_n\prod_{v=1}^{n}(z-z_v)$, $a_n\neq 0$, be a polynomial of degree n. If $|z_v|\geqslant K_v\geqslant 1$, $1\leqslant v\leqslant n$, then*

$$\max_{|z|=1}|P_n'(z)|\leqslant n\left(\frac{\sum_{v=1}^{n}(1/(K_v-1))}{\sum_{v=1}^{n}((K_v+1)/(K_v-1))}\right)\max_{|z|=1}|P_n(z)|$$

$$=\frac{n}{2}\left\{1-\frac{1}{1+(2/n)\sum_{v=1}^{n}(1/(K_v-1))}\right\}\max_{|z|=1}|P_n(z)|. \quad (1.5)$$

In the case $K_v=K\geqslant 1$ for $1\leqslant v\leqslant n$, the result is best possible and the equality holds for the polynomial $p(z)=(z+K)^n$, $K\geqslant 1$.

2. LEMMAS

We will need the following lemmas.

LEMMA 1. *If $P_n(z)$ is a polynomial of degree n, then for $|z|=1$,*

$$|P_n'(z)|+|Q_n'(z)|\leqslant n\max_{|z|=1}|P_n(z)|. \quad (2.1)$$

Here and elsewhere $Q_n(z)$ stands for the polynomial $z^n\{\overline{P_n(1/\bar{z})}\}$.

This lemma is a special case of a result due to Govil and Rahman [3, Lemma 10].

LEMMA 2. *Let $P_n(z)=a_n\prod_{v=1}^{n}(z-z_v)$, $a_n\neq 0$, be a polynomial of degree n. If $|z_v|\geqslant K_v\geqslant 1$, $1\leqslant v\leqslant n$, and $Q_n(z)=z^n\{\overline{P_n(1/\bar{z})}\}$, then for $|z|=1$,*

$$|Q_n'(z)|/|P_n'(z)|\geqslant t_0, \quad (2.2)$$

where $t_0=\{1+n/\sum_{v=1}^{n}(1/(K_v-1))\}\geqslant 1$.

Proof. Note that from Govil and Labelle [2, (3.5) on p. 500] we have for $|z|=1$,

$$|Q_n'(z)|/|P_n'(z)|\geqslant\left(\sum_{v=1}^{n}\frac{|z_v|}{|z_v|-1}\right)\Big/\left(\sum_{v=1}^{n}\frac{1}{|z_v|-1}\right),$$

$$=1+\frac{n}{\sum_{v=1}^{n}(1/(|z_v|-1))},$$

and since the expression on the right hand side is clearly a non-decreasing function in each of the variables $|z_1|, |z_2|, ..., |z_n|$, we get

$$|Q_n'(z)|/|P_n'(z)| \geqslant 1 + \frac{n}{\sum_{v=1}^{n} (1/(K_v - 1))}$$
$$= t_0.$$

It is obvious that $t_0 \geqslant 1$, and the lemma follows.

LEMMA 3. *Let $\mathscr{P}_n$ denote the linear space of polynomials*

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

of degree at most n with complex coefficients normed by $\|P_n\| = \max_{0 \leqslant \theta < 2\pi} |P_n(e^{i\theta})|$. Define the linear functional L on $\mathscr{P}_n$ as

$$L: P_n \rightarrow l_0 a_0 + l_1 a_1 + \cdots + l_n a_n,$$

where the l_i are complex numbers. If the norm of the functional is N, then

$$\int_0^{2\pi} \Theta\left(\frac{|\sum_{k=0}^{n} l_k a_k e^{ik\theta}|}{N}\right) d\theta \leqslant \int_0^{2\pi} \Theta\left(\left|\sum_{k=0}^{n} a_k e^{ik\theta}\right|\right) d\theta, \qquad (2.3)$$

where $\Theta(t)$ is a non-decreasing convex function of t.

The above lemma is due to Rahman [5, Lemma 3].

3. PROOF OF THE THEOREM

The proof of the theorem is on the lines of the proof given by Rahman [5] of De Bruijn's theorem; however, for the sake of completeness we present the brief outlines.

Let $M = \max_{|z|=1} |P_n(z)|$. Since $Q_n(z) = z^n\{\overline{P_n(1/\bar{z})}\}$, hence by Lemma 1,

$$\left|\frac{d}{d\theta} P_n(e^{i\theta})\right| - \left|\frac{d}{d\theta} P_n(e^{i\theta}) - inP_n(e^{i\theta})\right| \leqslant Mn, \qquad (3.1)$$

which gives for every α, $0 \leqslant \alpha < 2\pi$,

$$\left|\frac{d}{d\theta} P_n(e^{i\theta}) + e^{i\alpha}\left\{\frac{d}{d\theta} P_n(e^{i\theta}) - inP_n(e^{i\theta})\right\}\right| \leqslant Mn,$$

which is equivalent to

$$\left|(e^{i\alpha} + 1)\frac{d}{d\theta} P_n(e^{i\theta}) - ine^{i\alpha} P_n(e^{i\theta})\right| \leqslant Mn.$$

Thus the norm, N, of the bounded linear functional L,

$$L\colon P_n \to \left[(e^{i\alpha}+1)\frac{d}{d\theta}P_n(e^{i\theta}) - ine^{i\alpha}P_n(e^{i\theta}) \right]_{\theta=0} \tag{3.2}$$

satisfies $N \leqslant n$ and hence by Lemma 3, for every $p \geqslant 1$,

$$\int_0^{2\pi} \left| \frac{d}{d\theta}P_n(e^{i\theta}) + e^{i\alpha}\left\{ \frac{d}{d\theta}P_n(e^{i\theta}) - inP_n(e^{i\theta}) \right\} \right|^p d\theta \leqslant n^p \int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta,$$

which gives

$$\int_0^{2\pi}\int_0^{2\pi} \left| \frac{d}{d\theta}P_n(e^{i\theta}) + e^{i\alpha}\left\{ \frac{d}{d\theta}P_n(e^{i\theta}) - inP_n(e^{i\theta}) \right\} \right|^p d\theta \, d\alpha$$

$$\leqslant 2\pi n^p \int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta. \tag{3.3}$$

Note that

$$\int_0^{2\pi}\int_0^{2\pi} \left| \frac{d}{d\theta}P_n(e^{i\theta}) + e^{i\alpha}\left\{ \frac{d}{d\theta}P_n(e^{i\theta}) - inP_n(e^{i\theta}) \right\} \right|^p d\theta \, d\alpha$$

$$= \int_0^{2\pi} \left| \frac{d}{d\theta}P_n(e^{i\theta}) \right|^p \int_0^{2\pi} \left| 1 + e^{i\alpha}\frac{\{(d/d\theta)P_n(e^{i\theta}) - inP_n(e^{i\theta})\}}{(d/d\theta)P_n(e^{i\theta})} \right|^p d\alpha \, d\theta$$

$$= \int_0^{2\pi} \left| \frac{d}{d\theta}P_n(e^{i\theta}) \right|^p \int_0^{2\pi} \left| 1 + e^{i\alpha}\left| \frac{(d/d\theta)P_n(e^{i\theta}) - inP_n(e^{i\theta})}{(d/d\theta)P_n(e^{i\theta})} \right| \right|^p d\alpha \, d\theta$$

$$= \int_0^{2\pi} \left| \frac{d}{d\theta}P_n(e^{i\theta}) \right|^p \int_0^{2\pi} \left| e^{i\alpha} + \left| \frac{(d/d\theta)P_n(e^{i\theta}) - inP_n(e^{i\theta})}{(d/d\theta)P_n(e^{i\theta})} \right| \right|^p d\alpha \, d\theta$$

$$= \int_0^{2\pi} \left| \frac{d}{d\theta}P_n(e^{i\theta}) \right|^p \int_0^{2\pi} \left| e^{i\alpha} + \left| \frac{Q_n'(e^{i\theta})}{P_n'(e^{i\theta})} \right| \right|^p d\alpha \, d\theta$$

$$\geqslant \int_0^{2\pi} \left| \frac{d}{d\theta}P_n(e^{i\theta}) \right|^p \int_0^{2\pi} |t_0 + e^{i\alpha}|^p \, d\alpha \, d\theta, \qquad \text{by Lemma 2,}$$

and this in view of (3.3) gives

$$\int_0^{2\pi} |P_n'(e^{i\theta})|^p \, d\theta \int_0^{2\pi} |t_0 + e^{i\alpha}|^p \, d\alpha \leqslant 2\pi n^p \int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta, \tag{3.4}$$

from which the theorem follows.

Acknowledgments

The authors are thankful to the referee and to Dr. Paul Deierman for their suggestions.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium