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GROWTH OF POLYNOMIALS NOT VANISHING INSIDE A CIRCLE

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Abstract: A well-known theorem of Ankeny and Rivlin states that if $p(z)$ is a polynomial of degree n, $p(z) \neq 0$ for $|z| < 1$, then $\max_{|z|=R>1} |p(z)| \leq (\frac{R^n+1}{2}) \max_{|z|=1} |p(z)|$. In this paper we generalize and sharpen this, and some other results in this direction.

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1. Introduction and Statement of Results

Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n, and let

$$\|p\| = \max_{|z|=1} |p(z)|, \quad M(p,R) = \max_{|z|=R} |p(z)|.$$

For a polynomial, $p(z) = \sum_{v=0}^{n} a_v z^v$, of degree n, it is well-known and is a simple consequence of maximum modulus principle (see [12] or [10, Volume 1, p. 137]) that for $R \geq 1$,

$$M(p,R) \leq R^n \|p\|, \tag{1.1}$$

with equality holding for $p(z) = \lambda z^n$, λ being a complex number. For a

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polynomial of degree n, not vanishing in $|z| < 1$, Ankeny and Rivlin [1] proved that for $R \geq 1$,

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\|. \tag{1.2}$$

The inequality (1.2) becomes equality for $p(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

Govil [5] observed that since the equality in (1.2) holds only for polynomials $p(z) = \lambda + \mu z^n$, $|\lambda| = |\mu|$, which satisfy

$$|\text{coefficient of } z^n| = \frac{1}{2}\|p\|, \tag{1.3}$$

it should be possible to improve upon the bound in (1.2) for polynomials not satisfying (1.3), and therefore in this connection he proved the following refinement of (1.2).

Theorem A. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n and $p(z) \neq 0$ in $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\| - \frac{n}{2}\left(\frac{\|p\|^2 - 4|a_n|^2}{\|p\|}\right) \times \left\{\frac{(R-1)\|p\|}{\|p\|+2|a_n|} - \ln\left(1+\frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right)\right\}. \tag{1.4}$$

The above inequality becomes equality for the polynomial $p(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

This result of Govil [5] was sharpened by Dewan and Bhat [4] who proved that under the hypotheses of Theorem A, the inequality (1.4) can in fact be replaced by a sharper inequality

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\| - \left(\frac{R^n-1}{2}\right)m - \frac{n}{2}\left(\frac{(\|p\|-m)^2 - 4|a_n|^2}{(\|p\|-m)}\right) \times \left\{\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+2|a_n|} - \ln\left(1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+2|a_n|}\right)\right\}, \tag{1.5}$$

where $m = \min_{|z|=1} |p(z)|$. The result is best possible and equality holds for the polynomial $p(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

The above result of Dewan and Bhat [4] was generalized by Govil and Nyuydinkong [7], who proved the following theorem.

Theorem B. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n and $p(z) \neq 0$ in $|z| < K$, $K \geq 1$, then for $R \geq 1$,*

$$M(p,R) \leq \Big(\frac{R^n + K}{1+K}\Big)\|p\| - \Big(\frac{R^n - 1}{1+K}\Big)m$$

$$-\frac{n}{1+K}\left(\frac{(\|p\|-m)^2 - (1+K)^2|a_n|^2}{(\|p\|-m)}\right) \times \left\{\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+K)|a_n|}\right.$$

$$\left. - \ln\left(1 + \frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+K)|a_n|}\right)\right\},$$

where $m = \min_{|z|=K} |p(z)|$.

In this paper, we prove the following generalization of Theorem B.

Theorem. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $1 \leq t \leq n$, is a polynomial of degree n and $p(z) \neq 0$ in $|z| < K$, $K \geq 1$, then for $R \geq 1$,*

$$M(p,R) \leq \Big(\frac{R^n + K^t}{1+K^t}\Big)\|p\| - \Big(\frac{R^n - 1}{1+K^t}\Big)m$$

$$-\frac{n}{1+K^t}\left(\frac{(\|p\|-m)^2 - (1+K^t)^2|a_n|^2}{(\|p\|-m)}\right) \times \left\{\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+K^t)|a_n|}\right.$$

$$\left. - \ln\left(1 + \frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+K^t)|a_n|}\right)\right\},$$

where $m = \min_{|z|=K} |p(z)|$.

Clearly, for $t = 1$, the above theorem gives Theorem B due to Govil and Nyuydinkong [7], which for $K = 1$ reduces to (1.5) due to Dewan and Bhat [4]. Since $(\|p\| - m)^2 - (1 + K^t)^2|a_n|^2 \geq 0$ (see Lemma 5) and $\ln(1 + x) < x$, for $x > 0$, our above theorem, in particular, gives the corollary.

Corollary. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $1 \leq t \leq n$, is a polynomial of degree n and $p(z) \neq 0$ in $|z| < K$, $K \geq 1$, then for $R \geq 1$,*

$$M(p,R) \leq \Big(\frac{R^n + K^t}{1+K^t}\Big)\|p\| - \Big(\frac{R^n - 1}{1+K^t}\Big)m,$$

where $m = \min_{|z|=K} |p(z)|$.

For $t = 1$, the above corollary clearly generalizes and sharpens the inequality (1.2) due to Ankeny and Rivlin [1]. It also includes as a special case (taking $t = 1$ and $K = 1$) the following result due to Aziz and Dawood [2], which is a sharpening of the inequality (1.2) due to Ankeny and Rivlin [1].

Theorem C. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *that does not vanish in* $|z| < 1$, *then for* $R \geq 1$,

$$M(p, R) \leq \Big(\frac{R^n + 1}{2}\Big)\|p\| - \Big(\frac{R^n - 1}{2}\Big)m,$$

where $m = \min_{|z|=1} |p(z)|$. *The above result is best possible and equality holds for the polynomial* $p(z) = \alpha z^n + \beta$, *where* $|\beta| \geq |\alpha|$.

2. Lemmas

We need the following lemmas.

Lemma 1. *Let* $f(z)$ *be analytic inside and on the circle* $|z| = 1$ *and let* $\|f\| = \max_{|z|=1} |f(z)|$. *If* $f(0) = a$, *where* $|a| < \|f\|$, *then for* $|z| < 1$,

$$|f(z)| \leq \Big(\frac{\|f\||z| + |a|}{\|f\| + |a||z|}\Big)\|f\|.$$

This is a well-known generalization of Schwarz Lemma (see for example [10, p. 167]).

Lemma 2. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n, *then for* $|z| = R \geq 1$,

$$|p(z)| \leq \Big(\frac{\|p\| + R|a_n|}{R\|p\| + |a_n|}\Big)\|p\|R^n.$$

The proof of this lemma follows easily by applying Lemma 1 to $T(z) = z^n p(\frac{1}{z})$ and noting that $\|T\| = \|p\|$ (see Rahman [11, Lemma 2] for details).

From Lemma 2, one immediately gets (see Govil [5, Lemma 3]).

Lemma 3. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then for $|z| = R \geq 1$,*

$$|p(z)| \leq R^n\Big(1 - \frac{(\|p\| - |a_n|)(R-1)}{(R\|p\| + |a_n|)}\Big)\|p\|.$$

Lemma 4. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $1 \leq t \leq n$, is a polynomial of degree n having no zeros in $|z| < K$, $K \geq 1$, then*

$$\|p'\| \leq \frac{n}{1+K^t}(\|p\| - m),$$

where $m = \min\limits_{|z|=K} |p(z)|$.

The above lemma is due to Govil [6, p. 629] and is of interest in itself, because it generalizes and sharpens results of Lax [8], Chan and Malik [3, Theorem 1], Malik [9, Theorem 1], and Aziz and Dawood [2, Theorem 2].

Lemma 5. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $1 \leq t \leq n$, is a polynomial of degree n having no zeros in $|z| < K$, $K \geq 1$, then*

$$|a_n| \leq \frac{1}{1+K^t}(\|p\| - m). \tag{2.1}$$

Proof. If $p(z) = \sum_{v=0}^{n} a_v z^v$, then $p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$. Hence Cauchy Inequality when applied to $p'(z)$ gives

$$|na_n| \leq \|p'\|. \tag{2.2}$$

On the other hand, by Lemma 4,

$$\|p'\| \leq \frac{n}{1+K^t}(\|p\| - m). \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$|na_n| \leq \frac{n}{1+K^t}(\|p\| - m),$$

from which (2.1) follows. □

Lemma 6. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n and $R \geq 1$, then*

$$\left(1 - \frac{(x - n|a_n|)(R-1)}{(Rx + n|a_n|)}\right)x$$

is an increasing function of x, for $x > 0$.

The above lemma which follows by the derivative test is also due to Govil [5, Lemma 5].

3. Proof of the Theorem

To prove the Theorem, first note that for each θ, $0 \leq \theta < 2\pi$, we have

$$p(Re^{i\theta}) - p(e^{i\theta}) = \int_1^R p'(re^{i\theta})e^{i\theta}dr.$$

Hence

$$\begin{aligned} |p(Re^{i\theta}) - p(e^{i\theta})| &\leq \int_1^R |p'(re^{i\theta})|dr \\ &\leq \int_1^R r^{n-1}\left(1 - \frac{(\|p'\| - n|a_n|)(r-1)}{(r\|p'\| + n|a_n|)}\right)\|p'\|dr, \quad (3.1) \end{aligned}$$

by applying Lemma 3 to $p'(z)$, which is a polynomial of degree $(n-1)$.

By Lemma 6, the integrand in (3.1) is an increasing function of $\|p'\|$, hence applying Lemma 4 to (3.1), we get for $0 \leq \theta < 2\pi$,

$$\begin{aligned} |p(Re^{i\theta}) - p(e^{i\theta})| \;\leq\; & \int_1^R r^{n-1}\left(1 - \frac{\{\frac{n}{1+K^t}(\|p\| - m) - n|a_n|\}(r-1)}{r\frac{n}{1+K^t}(\|p\| - m) + n|a_n|}\right) \\ & \times \frac{n}{1+K^t}(\|p\| - m)dr \end{aligned}$$

$$= \frac{n}{1+K^t}(\|p\| - m)\int_1^R r^{n-1}\left(1 - \frac{\{(\|p\| - m) - (1+K^t)|a_n|\}(r-1)}{r(\|p\| - m) + (1+K^t)|a_n|}\right)dr$$

$$\begin{aligned} = \frac{n}{1+K^t}(\|p\| - m)\int_1^R r^{n-1}dr - \frac{n}{1+K^t}\Big((\|p\| - m) - (1+K^t)|a_n|\Big) \\ \times \int_1^R \left(\frac{r^{n-1}(r-1)(\|p\| - m)}{r(\|p\| - m) + (1+K^t)|a_n|}\right)dr. \end{aligned}$$

Since by Lemma 5, $(\|p\| - m) - (1+K^t)|a_n| \geq 0$, we get for $0 \leq \theta \leq 2\pi$ and $R \geq 1$,

$$|p(Re^{i\theta}) - p(e^{i\theta})| \leq \frac{(R^n - 1)}{1+K^t}(\|p\| - m) - \frac{n}{1+K^t}\Big((\|p\| - m) - (1+K^t)|a_n|\Big)$$

$$\times \int_1^R \left(\frac{(r-1)(\|p\|-m)}{r(\|p\|-m)+(1+K^t)|a_n|} \right) dr$$

$$= \frac{(R^n-1)}{1+K^t}(\|p\|-m) - \frac{n}{1+K^t}\Big((\|p\|-m)-(1+K^t)|a_n|\Big)$$

$$\times \int_1^R \left(1 - \frac{(\|p\|-m)+(1+K^t)|a_n|}{r(\|p\|-m)+(1+K^t)|a_n|} \right) dr$$

$$= \frac{(R^n-1)}{1+K^t}(\|p\|-m) - \frac{n}{1+K^t}\Big((\|p\|-m)-(1+K^t)|a_n|\Big)$$

$$\times \left\{ (R-1) - \left(\frac{(\|p\|-m)+(1+K^t)|a_n|}{(\|p\|-m)} \right) \times \ln\left(\frac{R(\|p\|-m)+(1+K^t)|a_n|}{(\|p\|-m)+(1+K^t)|a_n|} \right) \right\}$$

$$= \frac{(R^n-1)}{1+K^t}(\|p\|-m) - \frac{n}{1+K^t}\Big((\|p\|-m)-(1+K^t)|a_n|\Big) \times \left(\frac{(\|p\|-m)+(1+K^t)|a_n|}{(\|p\|-m)} \right)$$

$$\times \left\{ \left(\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+K^t)|a_n|} \right) - \ln\left(\frac{R(\|p\|-m)+(1+K^t)|a_n|}{(\|p\|-m)+(1+K^t)|a_n|} \right) \right\}$$

$$= \frac{(R^n-1)}{1+K^t}(\|p\|-m) - \frac{n}{1+K^t}\Big(\frac{(\|p\|-m)^2-(1+K^t)^2|a_n|^2}{(\|p\|-m)} \Big)$$

$$\times \left\{ \left(\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+K^t)|a_n|} \right) - \ln\left(\frac{R(\|p\|-m)+(1+K^t)|a_n|}{(\|p\|-m)+(1+K^t)|a_n|} \right) \right\},$$

which clearly gives

$$M(p,R) \le \Big(\frac{R^n+K^t}{1+K^t} \Big) \|p\|$$

$$-\Big(\frac{R^n-1}{1+K^t} \Big) m - \frac{n}{1+K^t} \left(\frac{(\|p\|-m)^2-(1+K^t)^2|a_n|^2}{(\|p\|-m)} \right)$$

$$\times \left\{ \left(\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+K^t)|a_n|} \right) - \ln\left(1 + \frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+K^t)|a_n|} \right) \right\},$$

and the proof of the theorem is complete. □

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