EAST JOURNAL ON APPROXIMATIONS
Volume 10, Number 3 (2004), 301-312

SOME RESULTS CONCERNING RATE OF GROWTH OF POLYNOMIALS

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A well-known result due to Ankeny and Rivlin [2] states that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n satisfying $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|.$$

It was proposed by late Professor R. P. Boas, Jr. to obtain inequality analogous to this inequality for polynomials having no zeros in $|z| < K$, $K > 0$. In this paper, we obtain some results in this direction, by considering polynomials of the form $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$, which have no zeros in $|z| < K$, $K \geq 1$.

1. Introduction

Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be an algebraic polynomial of degree at most n, z being a complex variable, $\|p\| = \max_{|z|=1} |p(z)|$, and let $M(p, R) = \max_{|z|=R} |p(z)|$.

S. Bernstein used the Maximum Modulus Theorem to prove (see, for example, [14]):

Theorem A. *If p is a polynomial of degree n, then for $R \geq 1$*

$$M(p, R) \leq R^n \|p\|.$$

Theorem A is a best possible result and equality holds if and only if $p(z) = \lambda z^n$, $\lambda \in \mathbb{C}$. Thus if we consider polynomials p such that $p(0) \neq 0$, then the bound on $M(p, R)$ could be improved. Along this line, Ankeny and Rivlin [2] proved:

Theorem B. *If p is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$*

$$M(p,R) \leq \frac{R^n + 1}{2}\|p\|.$$

The above result is best possible and the equality holds for $p(z) = z^n + 1$.

As a refinement of the above result, Aziz and Dawood [3] proved

Theorem C. *If p is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$*

$$M(p,R) \leq \frac{R^n + 1}{2}\|p\| - \frac{R^n - 1}{2}\min_{|z|=1}|p(z)|.$$

The above Theorems B and C have been generalized and refined by Govil [7], by proving

Theorem D. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$, is a polynomial of degree $n \geq 2$, $p(z) \neq 0$ for $|z| < K$, $K \geq 1$, and if $m = \min_{|z|=K}|p(z)|$, then for $R \geq 1$,*

$$\begin{aligned} M(p,R) \leq & \left(\frac{R^n + K^t}{1+K^t}\right)\|p\| - \left(\frac{R^n - 1}{1+K^t}\right)m \\ & - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|p'(0)|, \end{aligned} \tag{1.1}$$

if $n > 2$, and

$$M(p,R) \leq \left(\frac{R^n + K^t}{1+K^t}\right)\|p\| - \left(\frac{R^n - 1}{1+K^t}\right)m - \frac{(R-1)^n}{2}|p'(0)|, \tag{1.2}$$

if $n = 2$.

In this paper, we first present a refinement of Theorem D, and so also of Theorems B and C. Our result in this direction is

Theorem 1. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$, is a polynomial of degree $n \geq 2$, $p(z) \neq 0$ for $|z| < K$, $K \geq 1$, and if $m = \min_{|z|=K}|p(z)|$, then for $R \geq 1$,*

$$\begin{aligned} M(p,R) \leq & \left(\frac{R^n + s_0}{1+s_0}\right)\|p\| - \left(\frac{R^n - 1}{1+s_0}\right)m \\ & - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|p'(0)| \end{aligned} \tag{1.3}$$

if $n > 2$, and

$$M(p,R) \leq \left(\frac{R^n + s_0}{1+s_0}\right)\|p\| - \left(\frac{R^n - 1}{1+s_0}\right)m - \frac{(R-1)^n}{2}|p'(0)|, \tag{1.4}$$

if $n = 2$. *Here*

$$s_0 = K^{t+1} \frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t+1} + 1}.$$

Theorem 1 sharpens Theorem D. To see this, first note that it follows, for example by the derivative test, that for every n and $R \geq 1$, the function $\left(\frac{R^n + x}{1 + x}\right) \|p\| - \left(\frac{R^n - 1}{1 + x}\right) m$, is a non-increasing function of x. If we combine this fact with Lemma 3, according to which $s_0 \geq K^t$ for $t \geq 1$, we get

$$\left(\frac{R^n + s_0}{1 + s_0}\right) \|p\| - \left(\frac{R^n - 1}{1 + s_0}\right) m \leq \left(\frac{R^n + K^t}{1 + K^t}\right) \|p\| - \left(\frac{R^n - 1}{1 + K^t}\right) m.$$

Thus the right-hand sides of (1.3) and (1.4) in Theorem 1, can not exceed the right-hand sides of (1.1) and (1.2) respectively in Theorem D, implying that Theorem 1 sharpens Theorem D.

Although, Theorem 1 in general gives a bound sharper than obtainable from Theorem D, in some cases the improvement can be considerably significant and this we show by means of the following example.

Example. Consider $p(z) = 1000 + z^2 + z^3 + z^4$. Clearly, here $t = 2$ and $n = 4$. We take $K = 5.5$, since we find numerically that $p(z) \neq 0$ for $|z| < 5.4483$. For this polynomial the bound for $M(p, 2)$ obtainable by Theorem D comes out to be 1530.3 while by Theorem 1 it comes out to be 1140.1, which is a significant improvement. Numerically, we find that for this polynomial $M(p, 2) \approx 1066.6$, which is quite close to the bound 1140.1, that we obtained by Theorem 1.

If in (1.3) and (1.4), we take $t = 1$, divide both sides by R^n and make $R \to \infty$, we get

Corollary 1. *If* $p(z) = a_0 + \sum_{v=1}^{n} a_v z^v$ *is a polynomial of degree* $n \geq 2$, $p(z) \neq 0$ *for* $|z| < K$, $K \geq 1$, *and if* $m = \min_{|z|=K} |p(z)|$, *then*

$$|a_n| + \frac{|a_1|}{n} \leq \frac{\|p\| - m}{1 + s_0}. \tag{1.5}$$

Since by Lemma 3, we have $s_0 \geq K^t$ for $t \geq 1$, the inequality (1.5) clearly sharpens inequality (1.8) of Govil [7] (p. 625), which for $K = 1$ sharpens the well-known inequality

$$|a_n| + \frac{|a_1|}{n} \leq \frac{\|p\|}{2},$$

that can be obtained by applying Visser's Inequality [15] to the polynomial $p'(z)$ and then combining it with with the well-known inequality of Lax [10], conjectured by Erdös.

For polynomials of any degree n, one can easily get from Theorem 1,

Corollary 2. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$ *is a polynomial of degree* n *and* $p(z) \neq 0$ *for* $|z| < K$, $K \geq 1$, *and if* $m = \min_{|z|=K} |p(z)|$, *then for* $R \geq 1$

$$M(p, R) \leq \left(\frac{R^n + s_0}{1 + s_0}\right) \|p\| - \left(\frac{R^n - 1}{1 + s_0}\right) m,$$

where s_0 *is as defined in Theorem* 1.

Since $f(x) = \frac{R^n + x}{1+x}\|p\| - \frac{R^n - 1}{1+x} m$ is a non-increasing function of x, and by Lemma 3, we have $s_0 \geq K^t \geq 1$, for $t \geq 1$, we immediately get

$$\frac{R^n + s_0}{1 + s_0}\|p\| - \frac{R^n - 1}{1 + s_0} m \leq \frac{R^n + 1}{2}\|p\| - \frac{R^n - 1}{2} m.$$

Thus, Corollary 2 is also a refinement of Theorem C, and so obviously of Theorem B.

If we do not have the knowledge of $\min_{|z|=K} |p(z)|$, we can use the following result, whose proof is similar to that of Theorem 1.

Theorem 2. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$ *is a polynomial of degree* $n \geq 2$ *and* $p(z) \neq 0$ *for* $|z| < K$, $K \geq 1$, *then for* $R \geq 1$:

$$M(p, R) \leq \left(\frac{R^n + s_1}{1 + s_1}\right) \|p\| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) |p'(0)|$$

if $n > 2$, *and*

$$M(p, R) \leq \left(\frac{R^n + s_1}{1 + s_1}\right) \|p\| - \frac{(R-1)^n}{2} |p'(0)|,$$

if $n = 2$, *where*

$$s_1 = K^{t+1} \frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|} K^{t+1} + 1}.$$

On applying Lemma 1 to the polynomial $\frac{p(Kz)}{a_0}$, which clearly satisfies its hypotheses, we get $\frac{t}{n}\left|\frac{a_t}{a_0}\right| K^t \leq 1$, which can easily be shown to be equivalent to $s_1 \geq K^t$. If we combine this with the fact that the function $\frac{R^n + x}{1+x}$ is a non-increasing function of x, Theorem 2 gives

Corollary 3. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$ is a polynomial of degree $n \geq 2$ and $p(z) \neq 0$ for $|z| < K$, $K \geq 1$, then for $R \geq 1$*

$$M(p,R) \leq \left(\frac{R^n + K^t}{1+K^t}\right) \|p\| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2}-1}{n-2}\right) |p'(0)|$$

when $n > 2$. When $n = 2$, we have

$$M(p,R) \leq \left(\frac{R^n + K^t}{1+K^t}\right) \|p\| - \frac{(R-1)^n}{2} |p'(0)|.$$

Of course with $K = 1$, Corollary 3 gives another refinement of Theorem B.

2. Lemmas

We need the following lemmas:

Lemma 1. *Let $p_n(z) = \prod_{\nu=1}^{n}(1 - z_\nu z)$ be a polynomial of degree n not vanishing in $|z| < 1$ and let $p_n'(0) = p_n''(0) = \cdots = p_n^{(l)}(0) = 0$. If $\Phi(z) = \{p_n(z)\}^\epsilon = \sum_{n=0}^{\infty} b_{k,\epsilon} z^k$, where $\epsilon = 1$ or -1, then*

$$|b_{k,\epsilon}| \leq n/k, \qquad (l+1 \leq k \leq 2l+1)$$

and

$$|b_{2l+2,1}| \leq \frac{n}{2(l+1)^2}(n+l-1), \quad |b_{2l+2,-1}| \leq \frac{n}{2(l+1)^2}(n+l+1).$$

The above result is due to Rahman and Stankiewicz (Theorem 2′, p. 180 of [13]).

Lemma 2. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n \geq 2$, $p(z) \neq 0$ in $|z| < K$, then $|p(z)| > m$ for $|z| < K$, and in particular*

$$|a_0| > m \tag{2.1}$$

where $m = \min_{|z|=K} |p(z)|$.

Proof. We can assume without loss of generality that $p(z)$ has no zeros on $|z| = K$, for otherwise the result holds trivially. Since $p(z)$, being a polynomial, is analytic in $|z| \leq K$ and has no zeros in $|z| \leq K$, by the Minimum Modulus Principle,

$$|p(z)| \geq m \text{ for } |z| \leq K$$

Since $p(z)$ is of degree $n \geq 2$, the above inequality implies that $|p(z)| > m$ for $|z| < K$, which in particular implies $|a_0| = |p(0)| > m$, which is (2.1). □

Lemma 3. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$, *is a polynomial of degree* $n \geq 2$, $p(z) \neq 0$ *for* $|z| < K$, $K \geq 1$, *and if* $m = \min_{|z|=K} |p(z)|$, *then*

$$s_0 = K^{t+1} \frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t+1} + 1} \geq K^t, \quad t \geq 1. \tag{2.2}$$

Proof. Without loss of generality we can assume $a_0 > 0$ for otherwise we can consider the polynomial $P(z) = e^{-\arg a_0} p(z)$, which clearly also has no zeros in $|z| < K$ and $M(P, R) = M(p, R)$. Since the polynomial $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$ is of degree $n \geq 2$ and $p(z) \neq 0$ for $|z| < K$, hence, by Lemma 2, the polynomial $p(z) - m \neq 0$ for $|z| < K$, implying that the polynomial $P(z) = p(Kz) - m \neq 0$ for $|z| < 1$. If we now apply Lemma 1 to the polynomial $\dfrac{P(z)}{a_0 - m}$, which clearly satisfies the hypotheses of Lemma 1, we get

$$\frac{|a_t| K^t}{a_0 - m} \leq \frac{n}{t}.$$

The above inequality is clearly equivalent to

$$\frac{t}{n} \left(\frac{|a_t| K^t}{|a_0| - m} \right) (K - 1) \leq K - 1,$$

which is equivalent to

$$\frac{t}{n} \left(\frac{|a_t| K^{t+1}}{a_0 - m} \right) + 1 \leq \frac{t}{n} \left(\frac{|a_t| K^t}{a_0 - m} \right) + K,$$

and from which (2.2) follows.

Lemma 4. *If* $p(z)$ *is a polynomial of degree* $n \geq 2$, *then for* $R \geq 1$,

$$M(p, R) \leq R^n \|p\| - (R^n - R^{n-2}) |p(0)|.$$

This result is due to Frappier, Rahman and Ruscheweyh (Theorem 2, p. 70 of [5]). In case, $p(z)$ is of degree 1, then it is clear that

$$M(p,R) = R\|p\| - (R-1)|p(0)|. \tag{2.3}$$

Lemma 5. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$, is a polynomial of degree n having no zeros in $|z| < K$, $K \geq 1$, then on $|z| = 1$,*

$$|p'(z)| \leq \frac{1}{K^{t+1}} \frac{(t/n)|a_t/a_0|K^{t+1}+1}{(t/n)|a_t/a_0|K^{t-1}+1} |q'(z)| \tag{2.4}$$

where $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

This lemma is implicit in Qazi (Lemma 1 of [12]), however, for the sake of completeness, we present a brief outline of its proof.

By a well-known theorem of Laguerre [9], if $p(z) \neq 0$ in $|z| < K$, then

$$np(z) - zp'(z) \neq -\zeta p'(z)$$

for $|z| < K$ and $|\zeta| < K$. The function

$$f(z) = \frac{Kp'(Kz)}{np(Kz) - Kzp'(Kz)}$$

is therefore analytic in $|z| \leq 1$ where it satisfies $|f(z)| \leq 1$. Further $f(0) = f'(0) = \cdots = f^{(t-2)}(0) = 0$ and $f^{(t-1)}(0) = (t/n)(a_t/a_0)K^t$, and therefore by the generalized form of Schwarz's lemma

$$|f(z)| \leq |z|^{t-1} \left(\frac{|z| + (t/n)|a_t/a_0|K^t}{(t/n)|a_t/a_0|K^t|z| + 1} \right)$$

for $|z| < 1$. In particular, this holds for $|z| < 1/K$, and so

$$|p'(z)| \leq \frac{1}{K^{t+1}} \frac{(t/n)|a_t/a_0|K^{t+1}+1}{(t/n)|a_t/a_0|K^{t-1}+1} |np(z) - zp'(z)|$$

for $|z| = 1$, from which (2.4) follows.

The inequality (2.4) is also clearly equivalent to

$$|q'(z)| \geq s_1 |p'(z)| \text{ on } |z| = 1 \tag{2.5}$$

where

$$s_1 = K^{t+1} \left(\frac{(t/n)|a_t/a_0|K^{t-1}+1}{(t/n)|a_t/a_0|K^{t+1}+1} \right).$$

Lemma 6. *The function*

$$s(x) = K^{t+1}\left(\frac{(t/n)(|a_t|/x)K^{t-1}+1}{(t/n)(|a_t|/x)K^{t+1}+1}\right)$$

is a non-decreasing function of x.

Proof. The proof follows by considering the first derivative of $s(x)$. □

Lemma 7. *If a polynomial* $p(z)$ *has no zeros in* $|z| < K$, $K \geq 1$, *then for every* λ, $|\lambda| < 1$

$$|q'(z)| \geq |\lambda| mn \text{ for } |z| = 1, \tag{2.6}$$

where $m = \min_{|z|=K} |p(z)|$ *and* $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$.

The proof of this lemma is implicit in Govil (Lemma 3 of [7]), however for the sake of completeness, we present its brief outlines. For this, first note that, we can assume without loss of generality that $p(z)$ has no zeros on $|z| = K$, for otherwise the result holds trivially.

Since $p(z)$ has no zeros in $|z| \leq K$, $K \geq 1$, the polynomial $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$ has all its zeros in $|z| < \frac{1}{K} \leq 1$ and hence by the Maximum Modulus Principle for unbounded domains:

$$\left|\frac{z^n}{q(z)}\right| \leq \frac{1/K^n}{\min_{|z|=1/K} |q(z)|} \text{ for } |z| \geq \frac{1}{K}$$

which is equivalent to

$$|q(z)| \geq K^n |z|^n \min_{|z|=1/K} |q(z)| \text{ for } |z| \geq \frac{1}{K}. \tag{2.7}$$

Note that, as is easy to verify,

$$\min_{|z|=1/K} |q(z)| = \frac{m}{K^n}$$

and this on combining with (2.7) gives

$$|q(z)| \geq m|z|^n \text{ for } |z| \geq \frac{1}{K}. \tag{2.8}$$

It follows from (2.8) that for every λ where $|\lambda| < 1$, the polynomial $q(z) - \lambda m z^n$ has all of its zeros in $|z| < 1/K$ and therefore by the Gauss-Lucas Theorem (p. 29 of [1]), the polynomial $q'(z) - \lambda mnz^{n-1}$ has all its zeros in $|z| < 1/K$, implying

$$|q'(z)| \geq |\lambda| mn|z|^{n-1} \text{ for } |z| \geq \frac{1}{K}. \tag{2.9}$$

Because $\frac{1}{K} \leq 1$, we get in particular from (2.9)

$$|q'(z)| \geq |\lambda| mn \text{ for } |z| = 1,$$

which is (2.6), and the proof of Lemma 7 is thus complete. □

Lemma 8. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$, *is a polynomial of degree* n *having no zeros in* $|z| < K$ *where* $K \geq 1$, *then for* $|z| = 1$,

$$|q'(z)| \geq s_0 |p'(z)| + mn, \tag{2.10}$$

where $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$, $m = \min_{|z|=K} |p(z)|$, *and*

$$s_0 = K^{t+1} \left(\frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t+1} + 1} \right).$$

Proof. The polynomial $p(z)$ has no zeros in $|z| < K$ where $K \geq 1$, and this by the Minimum Modulus Principle implies that for every λ with $|\lambda| < 1$ the polynomial $p(z) - \lambda m$ has also no zeros in $|z| < K$ where $K \geq 1$. Therefore applying (2.4) to the polynomial $p(z) - \lambda m$ we get on $|z| = 1$ that

$$|q'(z) - \overline{\lambda} mnz^{n-1}| \geq K^{t+1} \left(\frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0 - \lambda m|} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0 - \lambda m|} K^{t+1} + 1} \right) |p'(z)|. \tag{2.11}$$

Since for every λ, $|\lambda| < 1$ we have

$$|a_0 - \lambda m| \geq |a_0| - |\lambda| m \geq |a_0| - m \tag{2.12}$$

and $|a_0| > m$ by Lemma 3, we get on combining (2.11), (2.12) and Lemma 6 that for every λ where $|\lambda| < 1$

$$|q'(z) - \overline{\lambda} mnz^{n-1}| \geq K^{t+1} \left(\frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t+1} + 1} \right) |p'(z)| \tag{2.13}$$

on $|z| = 1$.

Now choosing the argument of λ so that on $|z| = 1$,

$$|q'(z) - \overline{\lambda} mnz^{n-1}| = |q'(z)| - |\lambda| mn \tag{2.14}$$

we get from (2.13) that on $|z| = 1$,

$$|q'(z)| \geq K^{t+1} \left(\frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|-m} K^{t+1} + 1} \right) |p'(z)| + |\lambda| mn. \tag{2.15}$$

The fact that the right-hand side of (2.14) is non-negative follows from Lemma 7. Lemma 8 now follows by making $|\lambda| \to 1$ in (2.15). □

Lemma 9. *If $p(z)$ is a polynomial of degree n, then on $|z| = 1$*

$$|p'(z)| + |q'(z)| \leq n\|p\|, \tag{2.16}$$

where $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

Lemma 9 is a special case of a result due to Govil and Rahman (Lemma 10 of [8]).

The following lemma is of independent interest because besides proving a generalization and refinement of the Erdös-Lax Theorem [10], it provides generalization and refinements of the results of Aziz and Dawood [3], Chan and Malik [4], Govil (p. 31 of [6]), Govil (Lemma 6 of [7]) and Malik [11].

Lemma 10. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$ is a polynomial of degree n having no zeros in $|z| < K$ where $K \geq 1$, then*

$$M(p', 1) \leq \frac{n}{1 + s_0}(\|p\| - m) \tag{2.17}$$

where $m = \min_{|z|=K} |p(z)|$ and

$$s_0 = K^{t+1} \left(\frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0| - m} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0| - m} K^{t+1} + 1} \right).$$

Proof. On combining (2.10) and (2.16), we get on $|z| = 1$

$$s_0|p'(z)| + mn + |p'(z)| \leq n\|p\|$$

from which (2.17) follows. □

Since $s_1 \geq K^t$, the following lemma which provides a generalization of Erdös-Lax Theorem [10] also sharpens results of Chan and Malik [4] and Malik [11].

Lemma 11. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$ is a polynomial of degree n having no zeros in $|z| < K$ where $K \geq 1$, then*

$$M(p', 1) \leq \frac{n}{1 + s_1}\|p\|$$

where

$$s_1 = K^{t+1} \left(\frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|} K^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0|} K^{t+1} + 1} \right).$$

Proof. On combining (2.5) and (2.16), we get on $|z| = 1$,

$$s_1|p'(z)| + |p'(z)| \leq n\|p\|,$$

and from which the result follows. □

Proof of Theorem 1. For $n > 2$ we have

$$\begin{aligned}
&|p(Re^{i\theta}) - p(e^{i\theta})| \\
&= \left|\int_1^R p'(re^{i\theta})e^{i\theta}\,dr\right| \leq \int_1^R |p'(re^{i\theta})|dr \\
&\leq \int_1^R (r^{n-1}M(p',1) - (r^{n-1} - r^{n-3})|p'(0)|)\,dr, \text{ by Lemma 4} \\
&= \frac{R^n - 1}{n} M(p',1) - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|p'(0)| \\
&\leq \frac{R^n - 1}{n}\left(\frac{n(\|p\| - m)}{1 + s_0}\right) - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|p'(0)|, \text{ by Lemma 10} \\
&= \frac{R^n - 1}{1 + s_0}\|p\| - \left(\frac{R^n - 1}{1 + s_0}\right)m - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|p'(0)|,
\end{aligned}$$

implying

$$|p(Re^{i\theta})| \leq \left(\frac{R^n + s_0}{1 + s_0}\right)\|p\| - \left(\frac{R^n - 1}{1 + s_0}\right)m - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|p'(0)|,$$

from which the theorem follows.

If $n = 2$, then $p'(z)$ is of degree 1, and so if we use (2.3) in place of Lemma 4, the result will follow, as above. □

Proof of Theorem 2. The proof of this result follows on the lines of the proof of Theorem 1, but on applying Lemma 11 instead of Lemma 10. We omit the details. □

Acknowledgements. The authors wish to express their gratitude to Dr. Xingchen Yuan for developing software which was used by the authors to create the example of Section 1.

References

[1] L. AHLFORS, *Complex Analysis*, McGraw-Hill, New York, 1979.

[2] N. C. ANKENY AND T. J. RIVLIN, On a theorem of S. Bernstein, *Pacific J. Math.* **5** (1955), 849–852.

[3] A. AZIZ AND M. DAWOOD, Inequalities for a polynomial and its derivative, *J. Approx. Theory* **53** (1988), 155–162.

[4] T. N. CHAN AND M. A. MALIK, On Erdös-Lax Theorem, *Proc. Indian Acad. Sci.* **92** (1983), 191–193.

[5] C. FRAPPIER, Q. I. RAHMAN AND ST. RUSCHEWEYH, New inequalities for polynomials, *Trans. Amer. Math. Soc.* **288** (1985), 69–99.

[6] N. K. GOVIL, Some inequalities for deriatives of polynomials, *J. Approx. Theory* **66** (1991), 29–35.

[7] N. K. GOVIL, On growth of polynomials, *J. of Inequal. & Appl.* **7** (2002), 623–631.

[8] N. K. GOVIL AND Q. I. RAHMAN, Functions of exponential type not vanishing in a half-plane and related polynomials, *Trans. Amer. Math. Soc.*, **137** (1969), 501–517.

[9] E. LAGUERRE, *Oeuvres*, Vol. **1**, Gauthier-Villars, Paris, 1898.

[10] P. D. LAX, Proof of a conjecture of P. Erdös, *Bull. Amer. Math. Soc.* **50** (1944), 509–513.

[11] M. A. MALIK, On the derivative of a polynomial, *J. London Math. Soc.* **1** (1969), 57–60.

[12] M. A. QAZI, On the maximum modulus of polynomials, *Proc. Amer. Math. Soc.* **115** (1992), 337–343.

[13] Q. I. RAHMAN AND J. STANKIEWICZ, Differential inequalities and local valency, *Pacific J. Math.* **54** (1974), 165–181.

[14] M. RIESZ, Über einen Satz des Herrn Serge Bernstein, *Acta Math.* **40** (1916), 337–347.

[15] C. VISSER, A simple proof of certain inequalities concerning polynomials, *Nederl. Akad. Wetensch. Proc.* **47** (1945), 276–281; *Indag. Math.*, **7** (1945), 81-86.

Received December 29, 2003

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