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Some Results on the Location of Zeros of Analytic Functions

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Abstract: The classical Eneström-Kakeya Theorem states that if $p(z)=\sum_{v=0}^{n}a_v z^v$ is a polynomial such that $0\le a_0\le a_1\le a_2\le\cdots\le a_n$,then all of the zeros of $p(z)$ lie in the region $|z|\le 1$ in the complex plane. Many generalizations of the Eneström-Kakeya Theorem exist which put various conditions on the coefficients of the polynomial (such as mononicity of the moduli of the coefficients). In this paper, we will introduce several results which put conditions on the coefficients of even powers of Z and on the coefficients of odd powers of Z. As a consequence, our results will be applicable to some analytic functions to which these related results are not applicable.

Keywords: analytic funetions; lecation of zeres mononicity

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1 Introduction

The classical Eneström-Kakeya Theorem states that if $p(z)=\sum_{v=0}^{n}a_v z^v$ is a polynomial such that $0\le a_0\le a_1\le a_2\le\cdots\le a_n$,then all of the zeros of $p(z)$ lie in the region $|z|\le 1$ in the complex plane. Many papers (cf.[2~4]) put various conditions of coefficients of a polynomial and obtained several results about the location of zeros of a polynomial by using mononicity of moduli of coefficients of a polynomial, others studied the location of zeros of a polynomial by using mononicity of real and imaginary parts of coefficients of a polynomial. In [1],the authors studied the location of zeros of an analytic function by putting conditions on moduli of coefficients of an analytic function. The following is main result discussed in theorem 6 in [1].

Theorem 1 Let $f(z)=\sum_{v=0}^{\infty}a_v z^v\ne 0$ be analytic in $|z|\le t$. If $|arg a_j|\le\alpha\le\frac{\pi}{2}$, $j\in\{0,1,2,3,\cdots\}$ and for finite nonnegative integer k,$|a_0|\le t|a_1|\le\cdots\le t^k a_k\ge t^{k+1}a_{k+1}\ge\cdots$ then $f(z)$ does not vanish in

$$|z|<\frac{t}{\left((2t^k|a_k/a_0|-1)\cos\alpha+\sin\alpha+\frac{2\sin\alpha}{|a_0|}\sum_1^{\infty}t^j|a_j|\right)}$$

It is will known that analytic functions such as sine,cosine, exponential and logarithm functions have many applications in the practical problems. Finding the locations of zeros of analytic functions is a widely useful topic in complex analysis, since the locations of zeros are the main

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properties of the analytic functions. In this paper, we will put conditions on the coefficients of even powers of Z and on the coefficients of odd powers of Z, and give the example to illustrate that our results will be applicable to some analytic functions to which these related results are not applicable.

2 The Main Results and Applications

Motivated by theorem 6 in [1], we put restriction on moduli of odd and even coefficients.

Theorem 2 If $P(z)=\sum_{\mu=0}^{\infty}a_\mu z^\mu$ is an analytic function in $|z|\le T$ and $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$ for $j\in\{0,1,2\cdots,\}$ and for some real β and some nonnegative integers k and l and some positive t such that $t\le T$

$$|a_0|\le|a_2|t^2\le|a_4|t^4\le\cdots\le|a_{2k}|t^{2k}\ge|a_{2k+2}|t^{2k+2}\ge\cdots\ge\cdots$$

$$|a_1|\le|a_3|t^2\le|a_5|t^4\le\cdots\le|a_{2l-1}|t^{2l-2}\ge|a_{2l+1}|t^{2l}\ge\cdots\ge\cdots$$

Then p(z) does not vanish in $|z|<R_1$ where

$$R_1=\min\left(\frac{t^3|a_0|}{M_1},t\right)$$

Here

$$M_1=-|a_0|t\cos\alpha+|a_1|t^2(1-\cos\alpha)+2(|a_{2k}|t^{2k+2}+|a_{2l-1}|t^{2l+1})\cos\alpha+\sum_{i=2}^{\infty}(|a_i|t^2+|a_{i-2}|)t^i\sin\alpha$$

Theorem 3 Let $P(z)=\sum_{\mu=0}^{\infty}a_\mu z^\mu$ be an analytic function in $|z|\le T$ with real coefficients such that

$$a_0\ge a_2\ge a_4\ge\cdots\ge$$

$$a_1\le a_3\le a_5\le\cdots\le$$

Then $p(z)$ does not vanish in $|z|<\min(R_1,t)$ where

$$R_1=\frac{t|a_0|}{M_1}$$

and

$$M_1=|a_0|+2|a_1|$$

We now apply this Theorem 3 to a specific analytic function.

Example 2.1. Consider $p(z)=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots=\cos z$ which is an analytic function in $|z|\le\infty$. Then according to Corollary 3, $p(z)$ does not vanish in $|z|<1$, that is, all zeros of $\cos z$ satisfy $|z|\ge 1$. We can not apply theorem 1 ([1]) to $\cos z$ because the coefficients of $\cos z$ do not satisfy the condition in theorem 1([1]).

In the following theorem, we obtain the following inequality by using generalization of Schwarz's inequality.

Theorem 4 If $P(z)=\sum_{\mu=0}^{\infty}a_\mu z^\mu$ is an analytic function in $|z|\le T$ and $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$ for $j\in\{0,1,2\cdots,\}$ and for some real β , and some nonnegative integers k and l, there exists some positive t such that $t\le T$ and

$$|a_0|\le|a_2|t^2\le|a_4|t^4\le\cdots\le|a_{2k}|t^{2k}\ge|a_{2k+2}|t^{2k+2}\ge\cdots\ge\cdots$$

$$|a_1|\le|a_3|t^2\le|a_5|t^4\le\cdots\le|a_{2l-1}|t^{2l-2}\ge|a_{2l+1}|t^{2l}\ge\cdots\ge\cdots$$

Then p(z) does not vanish in $|z|<R_1$ where

$$R_1=\min\left(\frac{t^4|a_0|(t^2|a_1|-M_1)+\{t^8|a_0|^2(|a_1|t^2-M_1)^2+4M_1^3t^4|a_0|\}^{\frac{1}{2}}}{2M_1^2},t\right)$$

Here

$$M_1=-|a_0|t\cos\alpha+|a_1|t^2(1-\cos\alpha)+2(|a_{2k}|t^{2k+2}+|a_{2l-1}|t^{2l+1})\cos\alpha+\sum_{i=2}^{\infty}(|a_i|t^2+|a_{i-2}|)t^i\sin\alpha$$

As inspired from [2] by putting restriction of real and imaginary parts of analytic function, we get the following theorem.

Theorem 5 Let $P(z)=\sum_{\mu=0}^{\infty}a_\mu z^\mu$ be an analytic function in $|z|\le T$ and $Re(a_j)=\alpha_j$, $Im(a_j)=\beta_j$ and for some nonnegative integers k,l,s and q , there exists a positive t such that $t\le T$ and

$$\alpha_0\le\alpha_2t^2\le\alpha_4t^4\le\cdots\le\alpha_{2k}t^{2k}\ge\alpha_{2k+2}t^{2k+2}\ge\cdots\ge\cdots$$

$$\alpha_1\le\alpha_3t^2\le\alpha_5t^4\le\cdots\le\alpha_{2l-1}t^{2l-2}\ge\alpha_{2l+1}t^{2l}\ge\cdots\ge\cdots$$

$$\beta_0\le\beta_2t^2\le\beta_4t^4\le\cdots\le\beta_{2s}t^{2s}\ge\beta_{2s+2}t^{2s+2}\ge\cdots\ge\cdots$$

$$\beta_1\le\beta_3t^2\le\beta_5t^4\le\cdots\le\beta_{2q-1}t^{2q-2}\ge\beta_{2q+1}t^{2q}\ge\cdots\ge\cdots$$

Then $p(z)$ does not vanish in $|z|\le R_1$ where

$$R_1=\min\left(\frac{t|a_0|}{M_1},t\right)$$

Here

$$M_1=(|\alpha_1|+|\beta_1|)t-(\alpha_1+\beta_1)t-(\alpha_0+\beta_0)+2[\alpha_{2k}t^{2k}+\alpha_{2l-1}t^{2l-1}+\beta_{2s}t^{2s}+\beta_{2q-1}t^{2q-1}]$$

The following inequality of analytic function is obtained by using generalization of Schwarz's inequality and mononicity of real and imaginary parts.

Theorem 6 Let $P(z)=\sum_{\mu=0}^{\infty}a_\mu z^\mu$ be an analytic function in $|z|\le T$ and $Re(a_j)=\alpha_j$, $Im(a_j)=\beta_j$ and for some nonnegative integers k ,l,s and q , there exists a positive t such that $t\le T$ and

$$\alpha_0\le\alpha_2t^2\le\alpha_4t^4\le\cdots\le\alpha_{2k}t^{2k}\ge\alpha_{2k+2}t^{2k+2}\ge\cdots\ge\cdots$$

$$\alpha_1\le\alpha_3t^2\le\alpha_5t^4\le\cdots\le\alpha_{2l-1}t^{2l-2}\ge\alpha_{2l+1}t^{2l}\ge\cdots\ge\cdots$$

$$\beta_0\le\beta_2t^2\le\beta_4t^4\le\cdots\le\beta_{2s}t^{2s}\ge\beta_{2s+2}t^{2s+2}\ge\cdots\ge\cdots$$

$$\beta_1 \le \beta_3 t^2 \le \beta_5 t^4 \le \cdots \le \beta_{2q-1} t^{2q-2} \ge \beta_{2q+1} t^{2q} \ge \cdots \ge \cdots$$

Then $p(z)$ does not vanish in $|z| \le R_1$ where

$$R_1 = \min\left(\frac{t^2|a_1|(|a_0| - M_1) + \{t^4|a_1|^2(|a_0| - M_1)^2 + 4M_1^3 t^2|a_0|\}^{\frac{1}{2}}}{2M_1^2}, t\right)$$

Here

$$M_1 = (|\alpha_1| + |\beta_1|)t - (\alpha_1 + \beta_1)t - (\alpha_0 + \beta_0) +$$
$$2[\alpha_{2k} t^{2k} + \alpha_{2l-1} t^{2l-1} + \beta_{2s} t^{2s} + \beta_{2q-1} t^{2q-1}]$$

3 Proof of the Main Results

We need the following lemma which is the generalization of Schwarz's inequality to prove the main theorems.

Lemma 1 Let $f(z)$ be analytic in $|z| < R$, $f(0) = 0$, $f'(z) = b$, and $|f(z)| \le M$ for $|z| = R$, then for $|z| \le R$

$$|f(z)| \le \frac{M|z|}{R^2} \frac{M|z| + R^2|b|}{M + |z||b|}$$

The following Lemma is due to Aziz and Mohammad [1].

Lemma 2 Let $P(z) = \sum_{\mu=0}^{n} a_\mu z^\mu$ be analytic in $|z| \le t$ such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$ for $j \in \{0, 1, 2 \cdots, n\}$ and for some real β, and positive t and nong ative integer k,

$$|a_0| \le |a_1|t \le |a_2|t^2 \le \cdots \le |a_k|t^k \ge |a_{k+1}|t^{k+1} \ge \cdots \ge |a_n|t^n$$

Then for $j \in \{1, 2 \cdots, n\}$

$$|ta_j - a_{j-1}| \le (t|a_j - |a_{j-1}|)\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha.$$

Proof of Theorem 2

Proof Consider the following analytic function $g(z)$

$$g(z) = (t^2 - z^2)p(z) = t^2 a_0 + a_1 t^2 z + \sum_{i=2}^{\infty}(a_i t^2 - a_{i-2})z^i = t^2 a_0 + G_1(z)$$

on $|z| = t$

$$|G_1(z)| \le |a_1|t^3 + \sum_{i=2}^{\infty}|a_i t^2 - a_{i-2}|t^i$$

By using Lemma 1 in the above, we obtain

$$|G_1(z)| \le |a_1|t^3 + \sum_{i=2}^{\infty}[(||a_i|t^2 - |a_{i-2}||)\cos\alpha + (|a_i|t^2 + |a_{i-2}|)\sin\alpha)]t^i$$
$$\le -|a_0|t^2\cos\alpha + |a_1|t^3(1 - \cos\alpha) + 2(|a_{2k}|t^{2k+2} + |a_{2l-1}|t^{2(l+1)})\cos\alpha +$$
$$+\sum_{i=2}^{\infty}(|a_i|t^2 + |a_{i-2}|)t^i \sin\alpha t^i = M_1$$

Then it follows from Schwarz's lemma, therefore

$$|G_1(z)| \le \frac{M_1|z|}{t} \quad for \quad |z| \le t$$

Which implies

$$\begin{aligned} |g(z)| &= |t^2a_0 + G_1(z)| \\ &\ge t^2|a_0| - |G_1(z)| \\ &\ge t^2|a_0| - \frac{M_1|z|}{t} \quad for\ |z| \le t \end{aligned}$$

Therefore, if $|z| \le R_1 = \min\{\frac{t^3|a_0|}{M_1}, t\}$, then $g(z) \ne 0$ and so $p(z) \ne 0$ that is, $p(z)$ does not vanish in $|z| \le R_1$

Proof of Theorem 4

Proof Consider the following analytic function

$$g(z) = (t^2 - z^2)p(z) = t^2a_0 + a_1t^2z + \sum_{i=2}^{\infty}(a_it^2 - a_{i-2})z^i = t^2a_0 + G_1(z)$$

on $|z| = t$

$$|G_1(z)| \le |a_1|t^3 + \sum_{i=2}^{\infty}|a_it^2 - a_{i-2}|t^i$$

By using the above Lemma 1, we obtain

$$\begin{aligned} |G_1(z)| &\le |a_1|t^3 + \sum_{i=2}^{\infty}[(||a_i|t^2 - |a_{i-2}||)\cos\alpha + (|a_i|t^2 + |a_{i-2}|)\sin\alpha)]t^i \\ &\le -|a_0|t^2\cos\alpha + |a_1|t^3(1-\cos\alpha) + 2(|a_{2k}|t^{2k+2} + |a_{2l-1}|t^{2l+1})\cos\alpha + \\ &\quad + \sum_{i=2}^{\infty}(|a_i|t^2 + |a_{i-2}|)t^i\sin\alpha t^i = M_1 \end{aligned}$$

Then it follows from Lemma 2, therefore

$$|G_1(z)| \le \frac{M_1|z|}{t^2}\frac{M_1|z| + t^4|a_1|}{M_1 + |z||a_1|t^2} \quad \text{for} \quad |z| \le t$$

Which implies

$$\begin{aligned} |g(z)| &= |t^2a_0 + G_1(z)| \\ &\ge t^2|a_0| - |G_1(z)| \\ &\ge t^2|a_0| - \frac{M_1|z|}{t^2}\frac{M_1|z| + t^4|a_1|}{M_1 + |z||a_1|t^2} \quad \text{for } |z| \le t \end{aligned}$$

Therefore, if $|z| \le R_1 = \min\left(\frac{t^4|a_1|(t^2-M_1) + \{t^8|a_1|^2(t^2-M_1)^2 + 4M_1^3t^4|a_0|\}^{\frac{1}{2}}}{2M_1^2}, t\right)$, then $g(z) \ne 0$ and so $p(z) \ne 0$ that is, $p(z)$ does not vanish in $|z| \le R_1$

Proof of Theorem 5

Proof We consider the following analytic function

$$g(z)=(t^2-z^2)p(z)=t^2a_0+a_1t^2z+\sum_{i=2}^{\infty}(a_it^2-a_{i-2})z^i=t^2a_0+G_1(z)$$

on $|z|=t$

$$|G_1(z)|\leq |a_1|t^3+\sum_{i=2}^{\infty}|a_it^2-a_{i-2}|t^i$$

$$\leq (|\alpha_1|+|\beta_1|)t^3+\sum_{i=2}^{\infty}(|\alpha_it^2-\alpha_{i-2}|t^i+|\beta_it^2-\beta_{i-2}|t^i)$$

$$\leq (|\alpha_1|+|\beta_1|)t^3-(\alpha_1+\beta_1)t^3-(\alpha_0+\beta_0)t^2+2[\alpha_{2k}t^{2k+2}+\alpha_{2l-1}t^{2l+1}+\beta_{2s}t^{2s+2}+\beta_{2q-1}t^{2q+1}]$$

$$=t^2M_1$$

We apply Schwarz's theorem [5, p.168] to $G_1(z)$,we get

$$|G_1(z)|\leq\frac{t^2M_1|z|}{t}=tM_1|z|,\quad for\quad |z|\leq t$$

Which implies

$$|g(z)|=|t^2a_0+G_1(z)|\geq t^2|a_0|-|G_1(z)|\geq t^2|a_0|-tM_1|z|\quad for|z|\leq t$$

Hence, if $|z|\leq R_1=\min\left(\frac{t|a_0|}{M_1},t\right)$,then $g(z)\neq 0$ and so $p(z)\neq 0$. that is, $p(z)$ does not vanish in $|z|\leq R_1$

Proof of Theorem 6

Proof Consider the following analytic function

$$g(z)=(t^2-z^2)p(z)=t^2a_0+a_1t^2z+\sum_{i=2}^{\infty}(a_it^2-a_{i-2})z^i=t^2a_0+G_1(z)$$

on $|z|=t$

$$|G_1(z)|\leq |a_1|t^3+\sum_{i=2}^{\infty}|a_it^2-a_{i-2}|t^i$$

$$\leq (|\alpha_1|+|\beta_1|)t^3+\sum_{i=2}^{\infty}(|\alpha_it^2-\alpha_{i-2}|t^i+|\beta_it^2-\beta_{i-2}|t^i)$$

$$\leq (|\alpha_1|+|\beta_1|)t^3-(\alpha_1+\beta_1)t^3-(\alpha_0+\beta_0)t^2+2[\alpha_{2k}t^{2k+2}+\alpha_{2l-1}t^{2l+1}+\beta_{2s}t^{2s+2}+\beta_{2q-1}t^{2q+1}]$$

$$=t^2M_1$$

We apply Lemma 2 to $G_1(z)$,we get

$$|G_1(z)|\leq\frac{M_1|z|(M_1|z|+t^2|a_1|}{M_1+|z||a_1|}),\quad \text{for}\quad |z|\leq t$$

Which implies

$$|g(z)|=|t^2a_0+G_1(z)|\geq t^2|a_0|-|G_1(z)|\geq t^2|a_0|-\frac{M_1|z|(M_1|z|+t^2|a_1|}{M_1+|z||a_1|}\quad \text{for}|z|\leq t$$

Hence, if $|z|\leq R_1=\min\left(\frac{t^2|a_1|(|a_0|-M_1)+\{t^4|a_1|^2(|a_0|-M_1)^2+4M_1^3t^2|a_0|\}^{\frac{1}{2}}}{2M_1^2},t\right)$,then $g(z)\neq 0$ and so $p(z)\neq 0$.

That is, $p(z)$ does not vanish in $|z|\leq R_1$

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解析函数零点位置的某些结果

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摘 要:古典的 Eneström-Kakeya 定理指出：如果$p(z)=\sum_{i=0}^{n}a_iz^i$ 是一个形如 $0\le a_0\le a_1\le a_2\le\cdots\le a_n$ 的多项式，则 $p(z)$ 的所有零点都落在 $|z|\le 1$ 的复平面区域内。多项式的系数加上多种限制条件后（如，系数模的单调性），就存在很多的 Eneström-Kakeya 推广的定理。本文中，将介绍当加上 Z 的偶次幂项和奇次幂项的系数条件限制后的一些结果。

关键词: 解析函数；零点位置；单调性

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Forced Oscillation of Systems of Nonlinear Neutral Parabolic Partial Functional Differential Equations

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Abstract: This paper studies the systems of nonlinear neutral parabolic partial functional differential equations with continuous distributed deviating arguments, Sufficient conditions are obtained for the forced oscillation of solutions of the systems.

Keywords: parabolic partial functional equations; nonlinear neutral type; continuous distributed deviating arguments; systems; the forced oscillation