1

Some Generalizations of Eneström-Kakeya Theorem

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Abstract. The Eneström-Kakeya Theorem states that if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial satisfying $0 < a_0 \le a_1 \le \ldots \le a_n$, then all the zeros of $p(z)$ lie in $|z| \le 1$. We present related results by considering polynomials with complex coefficients and by putting restrictions on the arguments and moduli of the coefficients.

Key words: Polynomials, zeros, Eneström-Kakeya Theorem.

1. INTRODUCTION AND STATEMENT OF RESULTS

The classical Eneström-Kakeya Theorem (see [6] for references) deals with the location of zeros in the complex plane:

Theorem 1.1. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial with real coefficients satisfying

$$0 < a_0 \le a_1 \le \ldots \le a_n,$$

then all the zeros of $p(z)$ lie in $|z| \le 1$.

There are several generalizations of the Eneström-Kakeya Theorem which weaken the hypotheses and are, therefore, applicable to a larger class of polynomials. Joyal, Labelle and Rahman [5] dropped the condition of positivity of the coefficients and proved:

Theorem 1.2. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial with real coefficients satisfying

$$a_0 \le a_1 \le \ldots \le a_n,$$

then all the zeros of $p(z)$ lie in $|z| \le \dfrac{a_n - a_0 + |a_0|}{|a_n|}$.

Of course, when $a_0 > 0$, Theorem 1.2 reduces to Theorem 1.1. A related result due to Govil and Rahman [3] concerns restrictions on the moduli and arguments of the coefficients and states:

Theorem 1.3. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$ for $j \in \{0, 1 \ldots n\}$ and for some real β, and

$$|a_0| \le |a_1| \le \ldots \le |a_n|,$$

then all the zeros of $p(z)$ lie in $|z| \le R$, where

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{\nu=0}^{n-1}|a_\nu|.$$

In this paper, we significantly weaken the condition of monotonicity on the moduli of the coefficients and prove:

Theorem 1.4. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$ for $j \in \{0, 1, \ldots, n\}$ and for some real β, and for some positive t and some non-negative integer K,

$$|a_0| \le t|a_1| \le \ldots \le t^K|a_K| \ge t^{K+1}|a_{K+1}| \ge \ldots \ge t^n|a_n|,$$

then all the zeros of $p(z)$ lie in $|z| \ge R$, where

$$R = \min\left\{\frac{|a_0|t}{\begin{array}{c}(2|a_K|t^K - |a_0|)\cos\alpha + |a_0|\sin\alpha + \\ 2\sin\alpha\sum_{v=1}^{n-1}|a_v|t^v + t^n|a_n|(1+\sin\alpha-\cos\alpha)\end{array}}, t\right\}.$$

The hypotheses of Theorem 1.4 were first introduced by Aziz and Mohammad in a study of the zeros of analytic functions (see Theorem 6 of [2]). If we are more restrictive on α, then we get the following:

Corollary 1.5. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{4}$ for $j \in \{0, 1, \ldots, n\}$ and for some real β, and for some positive t and some non-negative integer K,

$$|a_0| \le t|a_1| \le \ldots \le t^K|a_K| \ge t^{K+1}|a_{K+1}| \ge \ldots \ge t^n|a_n|,$$

then all the zeros of $p(z)$ lie in $|z| \ge R$, where

$$R = \frac{|a_0|t}{\begin{array}{c}(2|a_K|t^K - |a_0|)\cos\alpha + |a_0|\sin\alpha + \\ 2\sin\alpha\sum_{v=1}^{n-1}|a_v|t^v + t^n|a_n|(1+\sin\alpha-\cos\alpha)\end{array}}.$$

The following is a generalization of Theorem 1.3:

Theorem 1.6. If $q(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$

for $j \in \{0, 1, \dots n\}$ and for some real β, and if for some positive t and some non-negative integer K,

$$|a_n| \le t|a_{n-1}| \le \dots \le t^K|a_{n-K}| \ge t^{K+1}|a_{n-K-1}| \ge \dots \ge t^n|a_0|,$$

then all the zeros of $q(z)$ lie in $|z| \le R$, where

$$R = \max \left\{ \frac{\begin{array}{c}(2|a_{n-K}|t^K - |a_n|)\cos\alpha + |a_n|\sin\alpha \\ + 2\sin\alpha \sum_{v=1}^{n-1} |a_{n-v}|t^v + t^n|a_0|(1+\sin\alpha - \cos\alpha)\end{array}}{|a_n|t}, \frac{1}{t} \right\}.$$

The proof of Theorem 1.6 follows by applying Theorem 1.4 to $p(z) = z^n q\left(\frac{1}{z}\right)$.

Notice that with $t = 1$ and $K = 0$ in Theorem 1.6, we get:

Corollary 1.7. If $q(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$ for $j \in \{0, 1, \dots, n\}$ and for some real β, and

$$|a_n| \ge |a_{n-1}| \ge \dots \ge |a_0|,$$

then all the zeros of $q(z)$ lie in $|z| \le R$, where

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{v=1}^{n-1} |a_{n-v}| + \frac{|a_0|}{|a_n|}(1 + \sin\alpha - \cos\alpha).$$

Notice that, since $1 + \sin\alpha - \cos\alpha \le 2\sin\alpha$ when $0 \le \alpha \le \pi/2$, Corollary 1.7 is an improvement of Theorem 1.3 (and when $\alpha = 0$, both of these results reduce to Theorem 1.1), though Corollary 1.7 is inherent in the proof of Theorem 1.3 (see [3]).

We can extract further corollaries from Theorems 1.4 and 1.6 by choosing $K \in \{0, n\}$ and $t \in \{t, 1\}$. For example, with $K = 0$ in Theorem 1.4, we get:

Corollary 1.8. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is polynomial such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$ for $j \in \{0, 1, \dots, n\}$ *and for some real* β, *and for some positive* t,

$$|a_0| \ge t|a_1| \ge \dots \ge t^n|a_n|,$$

then all the zeros of $p(z)$ lie in $|z| \ge R$, where

$$R = \frac{|a_0|t}{\begin{array}{c}|a_0|(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{v=1}^{n-1}|a_v|t^v \\ + t^n|a_n|(1+\sin\alpha - \cos\alpha)\end{array}}.$$

The proof of Theorem 1.4 will employ Schwarz's Lemma. We can use a generalization of Schwarz's Lemma to produce a result which, although not as concise as Theorem 1.4, can produce bounds on the zero containing region of a polynomial which are better than those of Theorem 1.4 (as we shall see in Section 2). We will show:

Theorem 1.9. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$ for j $\in$ {0, 1, ..., n} and for some real β, and for some positive t and some non-negative integer K,

$$|a_0| \le t|a_1| \le \dots \le t^K|a_K| \ge t^{K+1}|a_{K+1}| \ge \dots \ge t^n|a_n|,$$

then all the zeros of $p(z)$ lie in $|z| \ge R$, where

$$R = \min\left\{\frac{-|b|t^2 (M - |a_0|t) + \{t^4 |b|^2 \times (M - |a_0|t)^2 + 4|a_0|M^3 t^3\}^{1/2}}{2M^2}, t\right\}$$

$$M = t|a_0| \left\{\left(2\left|\frac{a_K}{a_0}\right|t^K - 1\right)\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{v=0}^{n-1}|a_v|t^v + t^n\left|\frac{a_n}{a_0}\right|(1 + \sin\alpha - \cos\alpha)\right\}$$

$$b = a_0 - ta_1.$$

If we are more restrictive on the parameter α, we can get the following result which is more concise than Theorem 1.9:

Corollary 1.10. If $\sum_{v=0}^{n} a_v z^v$ is a polynomial such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{4}$ for $j \in$ {0, 1, ..., n} and for some real β, and for some positive t and some non-negative integer K,

$$|a_0| \le t|a_1| \le \dots \le t^K|a_K| \ge t^{K+1}|a_{K+1}| \ge \dots \ge t^n|a_n|,$$

then all the zeros of $p(z)$ lie in $|z| \ge R$, where

$$R = \frac{-|b|t^2 (M - |a_0|t) + \{t^4 |b|^2 (M - |a_0|t)^2 + 4|a_0|M^3 t^3\}^{1/2}}{2M^2}$$

and where M and b are as defined in Theorem 1.9.

Analogous to Theorem 1.6, we get the following by applying Theorem 1.9 to $p(z) = z^n q\left(\frac{1}{z}\right)$.

Theorem 1.11. If $q(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$ for $j \in$ {0, 1, ..., n} for some real β, and for some positive t and some non-negative integer K,

$$|a_n| \le t|a_{n-1}| \le \dots \le t^K|a_{n-K}| \ge t^{K+1}|a_{n-K-1}| \ge \dots \ge t^n|a_0|,$$

then all the zeros of $q(z)$ lie in $|z| \le R$, where

$$R = \max \left\{ \frac{2M^2}{-|b|t^2 (M - |a_n|t) + \{t^4 |b|^2 \times (M - |a_n|t)^2 + 4|a_n| M^3 t^3\}^{1/2}}, \frac{1}{t} \right\}$$

$$M = t|a_n| \left\{ \left(2\left|\frac{a_{n-K}}{a_0}\right| t^K - 1 \right) \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|} \sum_{v=0}^{n-1} |a_{n-v}| t^v + t^n \left|\frac{a_0}{a_n}\right| (1 + \sin\alpha - \cos\alpha) \right\}$$

$b = a_n - ta_{n-1}$.

Again, we can extract a more concise corollary from Theorem 1.11 by restricting $\alpha \le \frac{\pi}{4}$. Also, we can get several corollaries from Theorems 1.9 and 1.11 by choosing $K \in \{0, n\}$ and/or $t \in \{t, 1\}$.

2. AN EXAMPLE

The proof of Theorem 1.9 is similar to the proof of Theorem 1.4, but uses a generalization of Schwarz's Lemma whereas the proof of Theorem 1.4 uses Schwarz's Lemma. One is therefore lead to believe that Theorem 1.9 should give better results than Theorem 1.4. Due to the complicated nature of the parameters in these theorems, it's difficult to compare the results directly. However, we can give an example to show that Theorem 1.9 *can* give better bounds than does Theorem 1.4.

Consider the polynomial $p(z) = \frac{1}{10} + 2z + 4z^2 + 8z^3$. By Theorem 1.4 with $t = \frac{1}{2}$ and $K = 1$, we get that $p(z) \neq 0$ for $|z| < \frac{1}{38}$. By Theorem 1.9 with $t = \frac{1}{2}$ and $K = 1$, we get that $p(z) \neq 0$ for $|z| < .04826$. This is an improvement of Theorem 1.4 by a factor of about 3.36 (in terms of the area of the zero-free region).

3. LEMMAS

We need the following lemmas.

Lemma 3.1. Let $f(z) = \sum_{v=0}^{\infty} a_v z^v$ be analytic in $|z| \le t$ such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$ for $j \in \{0, 1, \ldots\}$ and for some real β, and if for positive t and some non-negative integer K,

$$|a_0| \le t|a_1| \le \ldots \le t^K|a_K| \ge t^{K+1}\,|a_K+1| \ge \ldots.$$

then for $j \in \{1, 2, \ldots\}$

$$|ta_j - a_{j-1}| \le |t|a_j| - |a_{j-1}||\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha.$$

Lemma 3.1 is due to Aziz and Mohammad [2].

Lemma 3.2. If $f(z)$ is analytic in $|z| \le R$, $f(0) = 0$, $f'(0) = b$ and $|f(z)| \le M$ for $|z| = R$, then for $|z| \le R$,

$$|f(z)| \le \frac{M|z|}{R^2}\frac{M|z| + R^2|b|}{M + |z||b|}.$$

Lemma 3.2 due to Govil, Rahman and Schmeisser [4].

Lemma 3.3. Let $a_0, a_1, \ldots, a_n$ be n complex numbers such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{4}$ for $j \in \{0, 1, \ldots, n\}$ and for some real β, and suppose for some positive t and some non-negative integer K,

$$|a_0| \le t|a_1| \le \ldots \le t^K|a_K| \ge t^{K+1}|a_{K+1}| \ge \ldots \ge t^n|a_n|.$$

Then $$\frac{|a_0|t}{\begin{gathered}(2|a_k|t^K - |a_0|)\cos\alpha + |a_0|\sin\alpha + \\ 2\sin\alpha\sum_{v=1}^{n-1}|a_v|t^v + t^n|a_n|(1 + \sin\alpha - \cos\alpha)\end{gathered}} \le t.$$

Proof. With $\alpha = 0$, the result becomes

$$\frac{|a_0|t}{2|a_K|t^k - |a_0|} = \frac{|a_0|t}{|a_K|t^K + (|a_K|t^K - |a_0|)} \le \frac{|a_0|t}{|a_K|t^K} \le t,$$

which is clearly true. Now let

$$f(\alpha) = (2|a_K|t^K - |a_0|)\cos\alpha + |a_0|\sin\alpha + 2\sin\alpha\sum_{v=1}^{n-1}|a_v|t^v$$
$$+\, t^n\,|a_n|(1 + \sin\alpha - \cos\alpha).$$

For $\alpha \in \left[0, \frac{\pi}{4}\right]$, we have

$$f'(\alpha) = (|a_0| - 2|a_K|t^K)\sin\alpha + |a_0|\cos\alpha + 2\cos\alpha\sum_{v=1}^{n-1}|a_v|t^v$$
$$+\, t^n|a_0|(\cos\alpha + \sin\alpha)$$
$$\ge (|a_0| - 2|a_K|t^K)\sin\alpha + 2|a_K|t^K\cos\alpha$$
$$\ge 2|a_K|t^K(\cos\alpha - \sin\alpha) \ge 0.$$

Therefore $f(\alpha)$ is an increasing function for $\alpha \in \left[0, \frac{\pi}{4}\right]$, and the result follows.

Lemma 3.4. Let $a_0, a_1, \ldots, a_n$ be n complex numbers such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{4}$ for $j \in \{0, 1, \ldots, n\}$ and for some real β, and suppose for some positive t and some non-negative integer K,

$$|a_0| \le t|a_1| \le \ldots \le t^K|a_K| \ge t^{K+1}|a_{K+1}| \ge \ldots \ge t^n|a_n|.$$

Then $$f(\alpha) = t|a_0|\left\{\left(2\left|\frac{a_K}{a_0}\right|t^K - 1\right)\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{v=1}^{n-1}|a_v|t^v\right.$$

$$\left. + t^n\left|\frac{a_n}{a_0}\right|(1 + \sin\alpha + \cos\alpha) - 1\right\} \ge 0.$$

Proof. First $f(0) = 2t|a_0|\left(\left|\frac{a_k}{a_0}\right|t^K - 1\right) \ge 0$. Now

$$f'(\alpha) = t|a_0|\left\{\left(1 - 2\left|\frac{a_k}{a_0}\right|t^K\right)\sin\alpha + \cos\alpha + \right.$$

$$\left.\frac{2\cos\alpha}{|a_0|}\sum_{v=1}^{n-1}|a_v|t^v + t^n\left|\frac{a_n}{a_0}\right|(\cos\alpha - \sin\alpha)\right\}$$

$$\ge t|a_0|\left\{\sin\alpha + 2\left|\frac{a_K}{a_0}\right|t^K(\cos\alpha - \sin\alpha) + t^n\left|\frac{a_n}{a_0}\right|(\cos\alpha - \sin\alpha)\right.$$

$$\left. + t^n\left|\frac{a_n}{a_0}\right|(\cos\alpha - \sin\alpha)\right\} \ge 0$$

since $\alpha \in \left[0, \frac{\pi}{4}\right]$. Therefore $f(\alpha) \ge 0$ for $\alpha \in \left[0, \frac{\pi}{4}\right]$.

4. PROOFS OF MAIN RESULTS

Proof of Theorem 1.4: Without loss of generality, we may assume $\beta = 0$. Consider

$$P_1(z) = (z - t)p(z) = -ta_0 + z\sum_{v=1}^{n}(a_{v-1} - ta_v)z^{v-1} + a_n z^{n+1}$$

$$\equiv -ta_0 + G_1(z).$$

Now since $|\arg a_j| \le \alpha \le \pi/2$ for $j \in \{0, 1, \ldots, n\}$, then by Lemma 3.1 for $|z| = t$,

$$|ta_j - a_{j-1}| \le |t|a_j| - |a_{j-1}||\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha.$$

Therefore for $|z| = t$,

$$|G_1(z)| \le t\sum_{v=1}^{n}|a_{v-1} - ta_v|t^{v-1} + t^{n+1}|a_n|$$

$$\leq t\left\{\sum_{v=1}^{n}\left|t|a_v|-|a_{v-1}|\right|t^{v-1}\cos\alpha+\right.$$

$$\left.\sum_{v=1}^{n}(t|a_v|+|a_{v-1}|)t^{v-1}\sin\alpha+t^n|a_n|\right\}$$

$$=t\left\{\cos\alpha\left[\sum_{v=1}^{K}(t|a_v|-|a_{v-1}|)t^{v-1}+\sum_{v=K+1}^{n}(|a_{v-1}|-t|a_v|)t^{v-1}\right]\right.$$

$$\left.+\sin\alpha\left[\sum_{v=1}^{n}t^v|a_v|+\sum_{v=1}^{n}|a_{v-1}|t^{v-1}\right]+t^n|a_n|\right\}$$

$$=t\left\{\cos\alpha\left[\sum_{v=1}^{K}|a_v|t^v-\sum_{v=1}^{K}|a_{v-1}|t^{v-1}\right.\right.$$

$$\left.+\sum_{v=K+1}^{n}|a_{v-1}|t^{v-1}-\sum_{v=K+1}^{n}|a_v|t^v\right]$$

$$\left.+\sin\alpha\left[2\sum_{v=1}^{n-1}|a_v|t^v+|a_0|+t^n|a_n|\right]+t^n|a_n|\right\}$$

$$=t\left\{\cos\alpha\left[t^K|a_K|-|a_0|+t^K|a_K|-t^n|a_n|\right]\right.$$

$$\left.+\sin\alpha\left[2\sum_{v=1}^{n-1}|a_v|t^v+|a_0|+t^n|a_n|\right]+t^n|a_n|\right\}$$

$$=t|a_0|\left\{\left(2\left|\frac{a_K}{a_0}\right|t^K-1\right)\cos\alpha+\sin\alpha+\frac{2\sin\alpha}{|a_0|}\sum_{v=1}^{n-1}|a_v|t^v\right.$$

$$\left.+t^n\left|\frac{a_n}{a_0}\right|(1+\sin\alpha-\cos\alpha)\right\}\equiv t|a_0|M \tag{4.1}$$

Now since $G_1(0)=0$, then it follows from Schwarz's Lemma (see page 135 of [1]) that

$$|G_1(z)|\leq|a_0|\,M|z| \text{ for } |z|\leq t.$$

So by (4.1)

$$\begin{aligned}|P_1(z)|&\geq t|a_0|-|G_1(z)|\\&\geq|a_0|\,[t-|z|M] \text{ for } |z|\leq t.\end{aligned}$$

Therefore, $P_1(z)>0$ if $|z|<\dfrac{t}{M}$ and $|z|\leq t$. The result follows.

Proof of Corollary 1.5: This follows from Theorem 1.4 and Lemma 3.3.

Proof of Theorem 1.9: Again, without loss of generality, we may assume $\beta = 0$. As in the proof of Theorem 1.4. Consider

$$P_1(z) = (z - t)p(z) \equiv -ta_0 + G_1(z).$$

As seen above,

$$|G_1(z)| \le t|a_0| \left\{ \left(2\left|\frac{a_K}{a_0}\right| t^K - 1 \right) \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|} \sum_{v=1}^{n-1} |a_v| t^v \right.$$

$$\left. + t^n \left|\frac{a_n}{a_0}\right| (1 + \sin\alpha - \cos\alpha) \right\}$$

$$\equiv M_1.$$

Since $G_1(z)$ is analytic for $|z| \le t$, $|G_1(z)| \le M_1$ on $|z| = t$, $G_1(0) = 0$, $G_1'(0) = a_0 - ta_1 = b$, then by Lemma 3.2 we have

$$|G_1(z)| \le \frac{M_1|z|}{t^2} \frac{M_1|z| + t^2|b|}{M_1 + |z||b|}$$

for $|z| \le$ t. Therefore

$$|P_1(z)| \ge t|a_0| - |G_1(z)|$$

$$\ge |a_0|t - \frac{M_1|z|}{t^2} \frac{M_1|z| + t^2|b|}{M_1 + |z||b|} \quad for |z| \le t$$

$$= |a_0|t - \frac{M_1^2|z|^2 + M_1|b|t^2|z|}{t^2(M_1 + |z||b|)}$$

$$= \frac{|a_0|t^3(M_1 + |z||b|) - M_1^2|z|^2 - M_1|b|t^2|z|}{t^2(M_1 + |z||b|)}$$

$$= \frac{-1}{t^2(M_1 + |z||b|)} (M_1^2|z|^2| + |b|t^2(M_1 - |a_0|t)|z| - |a_0|t^3M_1).$$

So $|P_1(z)| > 0$ if $|z| \le t$ and

$$|z| < \frac{-|b|t^2(M_1 - |a_0|t) + \{t^4|b|^2(M_1 - |a_0|t)^2 + 4|a_0|M_1^3t^3\}^{\frac{1}{2}}}{2M_1^2}.$$

So $|P_1(z)| > 0$ if $|z| < R$, and the result follows.

Proof of Corollary 1.10: Let M and b be as defined in Theorem 1.9. Notice that

$$\frac{-|b|t^2(M-|a_0|t)+\{t^4|b|^2(M-|a_0|t)^2+4|a_0|M^3t^3\}^{1/2}}{2M^2}\le t$$

if and only if $(M+t|b|)(M-|a_0|t)\ge 0$. Since $\alpha\in\left[0,\frac{\pi}{4}\right]$, we have from Lemma 3.4 that $M-|a_0|t\ge 0$ and the result follows.

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