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The Eneström–Kakeya Theorem for polynomials of a quaternionic variable

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Abstract

The well-known Eneström–Kakeya Theorem states that a polynomial with real, nonnegative, monotone increasing coefficients has all its complex zeros in the closed unit disk in the complex plane. In this paper, we extend this result by showing that all quaternionic zeros of such a polynomial lie in the unit sphere in the quaternions. We also extend related results from the complex to quaternionic setting.

MSC: primary 30E10; secondary 16K20

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1. Introduction

While studying the theory of pension funds in the 1890s, Gustav Eneström was lead to explore the zeros of a polynomial with real, positive, monotone coefficients. He proved the following [2].

Theorem 1 (*Eneström–Kakeya Theorem*). *If* $p(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *is a polynomial of degree* n *(where* z *is a complex variable) with real coefficients satisfying* $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$*, then all the zeros of* p *lie in* $|z| \leq 1$.

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Soichi Kakeya independently proved Theorem 1 and published his proof in English in 1912 [10]. Eneström later published a French translation of his earlier proof (which appeared in Swedish) in 1920 [3]. For these reasons, the result has become known as the "Eneström-Kakeya Theorem". For a detailed survey of the result and its generalizations, see [5].

An early generalization of Theorem 1 was due to Joyal, Labelle, and Rahman in 1967. They modified the Eneström–Kakeya Theorem by dropping the condition of nonnegative coefficients, as follows (see [9]).

Theorem 2. *If* $p(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *is a polynomial of degree* n *(where* z *is a complex variable) with real coefficients satisfying* $a_0 \le a_1 \le \cdots \le a_n$*, then all the zeros of* p *lie in* $|z| \le (|a_0| - a_0 + a_n)/|a_n|$.

Govil and Rahman presented a result applicable to polynomials with complex coefficients, as follows (see [8]).

Theorem 3. *If* $p(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *is a polynomial of degree* n *with complex coefficients satisfying* $|\arg a_\ell - \beta| \le \theta \le \pi/2$ *for some* β *and* θ *and for* $\ell = 0, 1, 2, \ldots, n$ *and* $|a_0| \le |a_1| \le \cdots \le |a_n|$*, then all the zeros of* p *lie in* $|z| \le \cos\theta + \sin\theta + \frac{2\sin\theta}{|a_n|}\sum_{\ell=0}^{n-1}|a_\ell|$.

In the same paper, Govil and Rahman gave a result for polynomials with complex coefficients and imposed a non-negativity and monotonicity condition on the coefficients, as follows (see [8]).

Theorem 4. *If* $p(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ *is a polynomial of degree* n *with complex coefficients where* $Re\, a_\ell = \alpha_\ell$ *and* $Im\, a_\ell = \beta_\ell$ *for* $\ell = 0, 1, 2, \ldots, n$*, satisfying* $0 \le \alpha_0 \le \alpha_1 \le \cdots \le \alpha_n$, $\alpha_n \ne 0$*, then all the zeros of* p *lie in* $|z| \le 1 + \frac{2}{\alpha_n}\sum_{\ell=0}^{n}|\beta_\ell|$.

2. Background

With the interpretation of the complex numbers as a two-dimensional "number system", Sir Rowan William Hamilton spent years trying to find a three-dimensional number system. He failed at this, but famously succeeded in finding a four-dimensional number system on October 16, 1843. This number system is the quaternions which we denote as $\mathbb{H}$ in honor of Hamilton. We use the standard notation $\mathbb{H} = \{\alpha + \beta i + \gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$, where i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1$. The quaternions are the standard example of a noncommutative division ring.

For $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$, the *real part* of q is α and β, γ, δ are the *imaginary parts* of q. The *conjugate* is $\overline{q} = \alpha - \beta i - \gamma j - \delta k$ and the *modulus* is $|q| = \sqrt{q\overline{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The modulus is then a norm on $\mathbb{H}$. For $r > 0$, we define the ball $B(0, r) = \{q \in \mathbb{H} \mid |q| < r\}$. We define the angle θ between two quaternions q_1 and q_2 by treating them as if they were vectors in $\mathbb{R}^4$. For $q_1 = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k$ and $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$, the *angle between* q_1 *and* q_2 is

$$\measuredangle(q_1, q_2) = \cos^{-1}\left(\frac{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 + \delta_1\delta_2}{|q_1||q_2|}\right).$$

We represent the indeterminate for a quaternionic polynomial as q. Without commutativity we are left with the polynomial aq^n and the polynomial $a_0 q a_1 q \cdots q a_n$, where $a = a_0 a_1 \cdots a_n$, as different. To alleviate this problem, we adopt the standard that polynomials have the indeterminate on the left and the coefficients on the right, so that we have the quaternionic polynomial $p_1(q) = \sum_{\ell=0}^{n} q^\ell a_n$. For such a p_1 and $p_2(q) = \sum_{\ell=0}^{m} q^\ell b_n$, the *regular product* is $(p_1 * p_2)(q) = \sum_{i=0,1,\ldots,n;\, j=0,1,\ldots,m} q^{i+j} a_i b_j$.

The absence of commutativity leads to a behavior of polynomials rather unlike their behavior in the real or complex settings. For example, a real or complex polynomial of degree n can have at most n (real or complex) zeros. In the quaternionic setting, the second degree polynomial q^2+1 has an infinite number of zeros; namely, any $q = \beta i + \gamma j + \delta k$ where $\beta^2+\gamma^2+\delta^2 = 1$. We denote the set of all such quaternions q as $\mathbb{S}$: $\mathbb{S} = \{\beta i + \gamma j + \delta k \mid \beta^2 + \gamma^2 + \delta^2 = 1\}$.

The following result concerning the roots of the regular product of two polynomials is from [11].

Theorem 5. *Let f and g be given quaternionic polynomials. Then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g(f(q_0)^{-1} q_0 f(q_0)) = 0$.*

An analytic theory of functions of a quaternionic variable has been developed recently. The next result illustrates the fundamental role played by the 2-sphere $\mathbb{S}$ in the zeros of quaternionic series, as well as polynomials (see [6]).

Theorem 6. *Let $\sum_{\ell=0}^{\infty} q^\ell a_\ell$ be a given quaternionic power series with radius of convergence R. Suppose that there exists $x_0, y_0 \in \mathbb{R}$ and $I, J \in \mathbb{S}$ with $I \neq J$ such that $\sum_{\ell=0}^{\infty}(x_0+y_0 I)^\ell a_\ell = 0$ and $\sum_{\ell=0}^{\infty}(x_0 + y_0 J)^\ell a_\ell = 0$. Then for all $L \in \mathbb{S}$ we have $\sum_{\ell=0}^{\infty}(x_0 + y_0 L)^\ell a_\ell = 0$.*

With this in mind, we see that we cannot use the degree of a polynomial as a bound on the number of zeros. However, Gentili and Struppa have given a definition for the multiplicity of the zeros of a polynomial such that the zeros counted by their multiplicity equal the degree of the polynomial (see [7]).

Gentili and Struppa also introduced a Maximum Modulus Theorem for regular functions [6]. Note, their class of regular functions includes convergent power series and polynomials.

Theorem 7 (*Maximum Modulus Theorem*). *Let $B = B(0, r)$ be a ball in $\mathbb{H}$ with center 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is constant on B.*

A number of results concerning polynomials have been extended from the complex setting to the quaternionic setting. In particular, Bernstein's inequality and some of its refinements have been extended; see Chapter 6 of [4]. In this paper we extend the Eneström–Kakeya Theorem from complex polynomials to quaternionic polynomials.

3. Statements of results

The proof of the Eneström–Kakeya Theorem only requires the Triangle Inequality for modulus and the Maximum Modulus Theorem. Since both of these hold in the quaternions, it is straightforward to extend the Eneström–Kakeya Theorem to functions of a quaternionic variable, as follows.

Theorem 8. *If $p(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|q| \leq 1$.*

We now show Theorem 8 is sharp. Consider the polynomial $p(q) = q^{n-1}+q^{n-2}+\cdots+q+1$. By Theorem 5, $p(q) * (q-1) = 0$ if and only if either (1) $p(q) = 0$, or (2) $p(q) \neq 0$ implies $p(q)^{-1}qp(q) - 1 = 0$. Notice that $p(q)^{-1}qp(q) - 1 = 0$ is equivalent to $p(q)^{-1}qp(q) = 1$ and, if $p(q) \neq 0$, this implies that $q = 1$. So the only zeros of $p(q) * (1-q)$ are $q = 1$ and the zeros of p. But $p(q) * (q-1) = q^n - 1$. Now we explore the roots of unity. For any $u \in \mathbb{S}$ (so $u^2 = -1$), we have $(\cos(2k\pi/n) + u\sin(2k\pi/n))^n = 1$ where $k \in \{0, 1, \ldots, n-1\}$ (this follows from De Moivre's Theorem; see, for example, [1]). Note, for $k = 0$ we get 1 as a root of unity. Moreover, if n is even and $k = n/2$ we also get -1 as a root of unity.

First, consider n odd. Then, we notice that $\cos((n-\ell)\pi/n) = \cos(\ell\pi/n)$ and $\sin((n-\ell)\pi/n) = -\sin(\ell\pi/n)$, so the pair of roots for $k = \ell$ and $k = n-\ell$ where $\ell \in \{1, 2, \ldots, (n-1)/2\}$) lie on the same sphere. The corresponding $(n-1)/2$ spheres are distinct since the real parts are distinct. By Theorem 6, all elements of these spheres are also roots of the polynomial. Therefore, for n odd the set of roots of $p(q) * (q-1)$ consists of 1 real root and $(n-1)/2$ isolated spheres, consistent with Gentili and Struppa's multiplicity theorem in [7].

Similarly, for n even the set of roots of $p(q) * (q-1)$ consists of two real roots (namely, 1 and -1) and $(n-2)/2$ isolated spheres (corresponding to $k \in \{1, 2, \ldots, (n-2)/2\}$ in the formula above). By Gentili and Struppa's multiplicity theorem in [7], this is all the roots of $p(q) * (q-1)$. The polynomial $p(q) = q^{n-1} + q^{n-2} + \cdots + q + 1$ has all coefficients real and equal and so it satisfies the hypotheses of the Eneström–Kakeya Theorem. So p has all roots on $|q| = 1$. This example shows that the bound in Theorem 8 is best possible.

The following is similar to Theorem 2 but instead of polynomials with monotone increasing real coefficients, it considers quaternionic polynomials with monotone increasing real parts and imaginary parts.

Theorem 9. *If $p(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable, where $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$ for $\ell = 0, 1, \ldots, n$, and satisfying $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n$, $\beta_0 \le \beta_1 \le \cdots \le \beta_n$, $\gamma_0 \le \gamma_1 \le \cdots \le \gamma_n$, $\delta_0 \le \delta_1 \le \cdots \le \delta_n$, then all the zeros of p lie in*

$$|q| \le \frac{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)}{|a_n|}.$$

Notice that if we take $\beta_\ell = \gamma_\ell = \delta_\ell = 0$ for $\ell = 0, 1, \ldots, n$ in Theorem 9 then we get Theorem 2 as a corollary.

We also extend Theorem 3 to quaternionic polynomials.

Theorem 10. *Let $p(z) = \sum_{\ell=0}^{n} q^\ell a_\ell$ be a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a nonzero quaternion and suppose $\measuredangle(a_\ell, b) \le \theta \le \pi/2$ for some θ and for $\ell = 0, 1, 2, \ldots, n$. Assume $|a_0| \le |a_1| \le \cdots \le |a_n|$. Then all the zeros of p lie in $|q| \le \cos\theta + \sin\theta + \frac{2\sin\theta}{|a_n|}\sum_{\ell=0}^{n-1} |a_\ell|$.*

In the terminology of vector spaces, the set $\{q \in \mathbb{H} \mid \measuredangle(q, b) = \pi/2\}$ is the "perp space" or "orthogonal complement" of the span of b (treating b as a vector) and $\{q \in \mathbb{H} \mid \measuredangle(q, b) \le \theta \le \pi/2\}$ is a "convex cone". If $a_\ell = \alpha_\ell + \beta_\ell i$, where $\alpha_\ell, \beta_\ell \in \mathbb{R}$, for $\ell = 0, 1, 2, \ldots, n$, then Theorem 10 implies Theorem 3.

Finally, we will extend Theorem 4 to quaternionic polynomials.

Theorem 11. *If $p(z) = \sum_{\ell=0}^{n} q^\ell a_\ell$ is a quaternionic polynomial of degree n where $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$ for $\ell = 0, 1, 2, \ldots, n$, satisfying $0 \le \alpha_0 \le \alpha_1 \le \cdots \le \alpha_n$, $\alpha_n \neq 0$, then all the zeros of p lie in $|z| \le 1 + \frac{2}{\alpha_n}\sum_{\ell=0}^{n}(|\beta_\ell| + |\gamma_\ell| + |\delta_\ell|)$.*

If we take $\gamma_\ell = \delta_\ell = 0$ for $\ell = 0, 1, \ldots, n$ in Theorem 11 then we get Theorem 4 as a corollary.

4. Proofs of results

We need the following for the proof of Theorem 10.

Lemma 12. *Let $q_1, q_2 \in \mathbb{H}$ where $q_1 = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k$ and $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$, $\measuredangle(q_1, q_1) = 2\theta' \leq 2\theta$, and $|q_1| \leq |q_2|$. Then*

$$|q_2 - q_1| \leq (|q_2| - |q_1|)\cos\theta + (|q_2| + |q_1|)\sin\theta.$$

Proof. Define $\vec{v}_1 = [\alpha_1, \beta_1, \delta_1, \gamma_1]$ and $\vec{v}_2 = [\alpha_2, \beta_2, \delta_2, \gamma_2]$ in $\mathbb{R}^4$. Then $\|\vec{v}_1\| = |q_1|$, $\|\vec{v}_2\| = |q_2|$. Let $2\theta'$ be the angle between $\vec{v}_1$ and $\vec{v}_2$. So

$$\begin{aligned}
\|\vec{v}_2 - \vec{v}_1\|^2 &= \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 - 2\|\vec{v}_1\|\|\vec{v}_2\|\cos 2\theta' \leq \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 - 2\|\vec{v}_1\|\|\vec{v}_2\|\cos 2\theta \\
&= (\|\vec{v}_1\| - \|\vec{v}_2\|)^2\cos^2\theta + (\|\vec{v}_1\| + \|\vec{v}_2\|)^2\sin^2\theta \\
&\leq (\|\vec{v}_1\| - \|\vec{v}_2\|)^2\cos^2\theta + 2(\|\vec{v}_1\| - \|\vec{v}_2\|)^2(\|\vec{v}_1\| + \|\vec{v}_2\|)^2\cos^2\theta\sin^2\theta \\
&\quad + (\|\vec{v}_1\| + \|\vec{v}_2\|)^2\sin^2\theta \\
&= ((\|\vec{v}_1\| - \|\vec{v}_2\|)\cos\theta + (\|\vec{v}_1\| + \|\vec{v}_2\|)\sin\theta)^2
\end{aligned}$$

and so

$$\|\vec{v}_2 - \vec{v}_1\| \leq (\|\vec{v}_2\| - \|\vec{v}_1\|)\cos\theta + (\|\vec{v}_1\| + \|\vec{v}_2\|)\sin\theta.$$

Since $\|\vec{v}_2 - \vec{v}_1\| = |q_2 - q_1|$, the claim holds. □

Proof of Theorem 8. Define f by the equation

$$p(q)*(1-q) = a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1}a_n = f(q) - q^{n+1}a_n.$$

By Theorem 5, $p(q) * (1 - q) = 0$ if and only if either $p(q) = 0$, or $p(q) \neq 0$ implies $p(q)^{-1}qp(q) - 1 = 0$. Notice that $p(q)^{-1}qp(q) - 1 = 0$ is equivalent to $p(q)^{-1}qp(q) = 1$ and, if $p(q) \neq 0$, this implies that $q = 1$. So the only zeros of $p(q) * (1 - q)$ are $q = 1$ and the zeros of p.

For $|q| = 1$, we have

$$\begin{aligned}
|f(q)| &= \left|a_0 + \sum_{\ell=1}^{n} q^n(a_\ell - a_{\ell-1})\right| \\
&\leq |a_0| + \sum_{\ell=1}^{n} |q^n(a_\ell - a_{\ell-1})| \\
&= |a_0| + \sum_{\ell=1}^{n} |a_\ell - a_{\ell-1}| \\
&= a_0 + \sum_{\ell=1}^{n} (a_\ell - a_{\ell-1}) \\
&= a_n.
\end{aligned}$$

Consider the function $q^n * f(1/q) = q^n * \sum_{\ell=0}^n q^{-\ell} a_\ell = \sum_{\ell=0}^n q^{n-\ell} a_\ell$. We have

$$\max_{|q|=1} |q^n * f(1/q)| = \max_{|q|=1} \left| q^n \sum_{\ell=0}^{n} q^{-\ell} a_\ell \right| = \max_{|q|=1} |f(1/q)| = \max_{|q|=1} |f(q)|.$$

So $q^n * f(1/q)$ has the same bound on $|q| = 1$ as f, namely $|q^n * f(1/q)| \leq a_n$ for $|q| = 1$. Since $q^n * f(1/q) = \sum_{\ell=0}^n q^{n-\ell} a_\ell = \sum_{\ell=0}^n q^n a_{n-\ell}$ is a polynomial and so is regular in $|q| \leq 1$, $|q^n * f(1/q)| = |q^n f(1/q)| \leq a_n$ for $|q| \leq 1$ by the Maximum Modulus Theorem (Theorem 7). Hence, $|f(1/q)| \leq a_n/|q|^n$ for $|q| \leq 1$. Replacing q with $1/q$, we see that

$$|f(q)| \leq a_n |q|^n \text{ for } |q| \geq 1. \tag{1}$$

Next, for $|q| \geq 1$ we have

$$\begin{aligned} |p(q) * (1-q)| &= |f(q) - q^{n+1} a_n| \\ &\geq a_n |q|^{n+1} - |f(q)| \\ &\geq a_n |q|^{n+1} - a_n |q|^n \text{ by (1)} \\ &= a_n |q|^n (|q| - 1). \end{aligned}$$

So if $|q| > 1$ then $|p(q) * (1-q)| > 0$ and $p(q) * (1-q) \neq 0$. Since the only zeros of $p(q) * (1-q)$ are $q = 1$ and the zeros of p, for $|q| > 1$ we have $p(q) \neq 0$. That is, all the zeros of p lie in $|q| \leq 1$ as claimed. □

The proofs of the three remaining theorems follow similar to that of the previous proof.

Proof of Theorem 9. Define f as in the proof of Theorem 8 as $f(q) = p(q)*(1-q)+q^{n+1}a_n$. For $|q| = 1$, we have

$$\begin{aligned} |f(q)| &= \left| a_0 + \sum_{\ell=1}^{n} q^n (a_\ell - a_{\ell-1}) \right| \\ &\leq |a_0| + \sum_{\ell=1}^{n} |a_\ell - a_{\ell-1}| \\ &= \sqrt{\alpha_0^2 + \beta_0^2 + \gamma_0^2 + \delta_0^2} \\ &+ \sum_{\ell=1}^{n} \sqrt{(\alpha_\ell - \alpha_{\ell-1})^2 + (\beta_\ell - \beta_{\ell-1})^2 + (\gamma_\ell - \gamma_{\ell-1})^2 + (\delta_\ell - \delta_{\ell-1})^2} \\ &\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| \\ &+ \sum_{\ell=1}^{n} (|\alpha_\ell - \alpha_{\ell-1}| + |\beta_\ell - \beta_{\ell-1}| + |\gamma_\ell - \gamma_{\ell-1}| + |\delta_\ell - \delta_{\ell-1}|) \\ &= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| - \alpha_0 - \beta_0 - \gamma_0 - \delta_0 + \alpha_n + \beta_n + \gamma_n + \delta_n. \end{aligned}$$

As in the proof of Theorem 8, for $|q| \geq 1$

$$\begin{aligned} |f(q)| \leq ((|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) \\ + (|\delta_0| - \delta_0 + \delta_n)) |q|^n. \end{aligned}$$

Next,

$$\begin{aligned}|p(q)*(1-q)| &\geq |a_n||q|^{n+1} - |f(q)| \\ &\geq |a_n||q|^{n+1} - ((|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) \\ &\quad + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n))\,|q|^n \\ &= (|a_n||q| - (|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) \\ &\quad + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n))\,|q|^n.\end{aligned}$$

So if

$$|q| > \frac{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)}{|a_n|}$$

(in which case $|q| \geq 1$) then $|p(q)*(1-q)| > 0$ and $p(q)*(1-q) \neq 0$. Since the only zeros of $p(q)*(1-q)$ are $q = 1$ and the zeros of p, for

$$|q| > \frac{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)}{|a_n|}$$

we have $p(q) \neq 0$. That is, all the zeros of p lie in

$$|q| \leq \frac{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)}{|a_n|},$$

as claimed. □

Proof of Theorem 10. Again let $f(q) = p(q)*(1-q) + q^{n+1}a_n$. For $|q| = 1$, we have

$$\begin{aligned}|f(q)| &= \left| a_0 + \sum_{\ell=1}^{n} q^n (a_\ell - a_{\ell-1}) \right| \\ &\leq |a_0| + \sum_{\ell=1}^{n} |a_\ell - a_{\ell-1}| \\ &\leq |a_0| + \sum_{\ell=1}^{n} ((|a_\ell| - |a_{\ell-1}|)\cos\theta + (|a_\ell| + |a_{\ell-1}|)\sin\theta) \text{ by Lemma 12} \\ &= |a_0|(1 - \cos\theta - \sin\theta) + |a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\ell=0}^{n-1} |a_\ell| \\ &\leq |a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\ell=0}^{n-1} |a_\ell|.\end{aligned}$$

As in the proof of Theorem 8,

$$|f(q)| \leq \left(|a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\ell=0}^{n-1} |a_\ell| \right) |q|^n \text{ for } |q| \geq 1.$$

Next,

$$\begin{aligned}|p(q)*(1-q)| &\geq |a_n||q|^{n+1} - |f(q)|\\ &\geq |a_n||q|^{n+1} - \left(|a_n|(\cos\theta+\sin\theta)+2\sin\theta\sum_{\ell=0}^{n-1}|a_\ell|\right)|q|^n\\ &= \left\{|a_n||q| - \left(|a_n|(\cos\theta+\sin\theta)+2\sin\theta\sum_{\ell=0}^{n-1}|a_\ell|\right)\right\}|q|^n.\end{aligned}$$

So if $|q| > \cos\theta+\sin\theta+\frac{2}{|a_n|}\sin\theta\sum_{\ell=0}^{n-1}|a_\ell|$ then $|p(q)*(1-q)|>0$ and $p(q)*(1-q)\neq 0$. Notice that

$$\cos\theta+\sin\theta+\frac{2}{|a_n|}\sin\theta\sum_{\ell=0}^{n-1}|a_\ell| \geq \cos\theta+\sin\theta \geq 1$$

since $\theta \in [0,\pi/2]$. So $|q| > \cos\theta+\sin\theta+\frac{2}{|a_n|}\sin\theta\sum_{\ell=0}^{n-1}|a_\ell|$ implies also that $|q| > 1$. Since the only zeros of $p(q)*(1-q)$ are $q = 1$ and the zeros of p, for $|q| > \cos\theta+\sin\theta+\frac{2}{|a_n|}\sin\theta\sum_{\ell=0}^{n-1}|a_\ell|$ we have $p(q)\neq 0$. That is, all the zeros of p lie in $|q| \leq \cos\theta+\sin\theta+\frac{2}{|a_n|}\sin\theta\sum_{\ell=0}^{n-1}|a_\ell|$, as claimed. □

Proof of Theorem 11. First, note that

$$\begin{aligned}|a_\ell - a_{\ell-1}| &= |(\alpha_\ell+\beta_\ell i+\gamma_\ell j+\delta_\ell k)-(\alpha_{\ell-1}+\beta_{\ell-1}i+\gamma_{\ell-1}j+\delta_{\ell-1}k)|\\ &\leq (\alpha_\ell-\alpha_{\ell-1})+|\beta_\ell|+|\beta_{\ell-1}|+|\gamma_\ell|+|\gamma_{\ell-1}|+|\delta_\ell|+|\delta_{\ell-1}|.\end{aligned}$$

Let

$$f(q) = p(q)*(1-q)-q^{n+1}\alpha_n = \sum_{\ell=1}^{n} q^\ell(a_\ell-a_{\ell-1})+a_0-q^{n+1}(\beta_n i+\gamma_n j+\delta_n k).$$

For $|q|=1$ we have

$$\begin{aligned}|f(q)| &= \left|\sum_{\ell=1}^{n} q^\ell(a_\ell-a_{\ell-1})+a_0-q^{n+1}(\beta_n i+\gamma_n j+\delta_n k)\right|\\ &\leq \sum_{\ell=1}^{n}(|a_\ell-a_{\ell-1}|)+|a_0|+|\beta_n|+|\gamma_n|+|\delta_n|\\ &\leq \sum_{\ell=1}^{n}(\alpha_\ell-\alpha_{\ell-1}+|\beta_\ell|+|\beta_{\ell-1}|+|\gamma_\ell|+|\gamma_{\ell-1}|+|\delta_\ell|+|\delta_{\ell-1}|)\\ &\quad+\alpha_0+|\beta_0|+|\gamma_0|+|\delta_0|+|\beta_n|+|\gamma_n|+|\delta_n|\\ &= \alpha_n+2\sum_{\ell=0}^{n}(|\beta_\ell|+|\gamma_\ell|+|\delta_\ell|),\end{aligned}$$

and so

$$|q^n * f(1/q)| = |q^n f(1/q)| \leq \alpha_n+2\sum_{\ell=0}^{n}(|\beta_\ell|+|\gamma_\ell|+|\delta_\ell|) \text{ for } |q|=1.$$

Then by the Maximum Modulus Theorem (Theorem 7),

$$|q^n f(1/q)| \le \alpha_n + 2\sum_{\ell=0}^{n}(|\beta_\ell| + |\gamma_\ell| + |\delta_\ell|) \text{ for } |q| \le 1.$$

Replacing q with $1/q$, we see that

$$|f(1/q)| \le |q|^n \left(\alpha_n + 2\sum_{\ell=0}^{n}(|\beta_\ell| + |\gamma_\ell| + |\delta_\ell|)\right) \text{ for } |q| \ge 1.$$

Next, for $|q| \ge 1$,

$$\begin{aligned} |p(q) * (1-q)| &= |f(q) + q^{n+1}\alpha_n| \\ &\ge |q^{n+1}\alpha_n| - |f(q)| \\ &\ge |q|^{n+1}\alpha_n - |q|^n \left(\alpha_n + 2\sum_{\ell=0}^{n}(|\beta_\ell| + |\gamma_\ell| + |\delta_\ell|)\right) \\ &= \left\{|q|\alpha_n - \left(\alpha_n + 2\sum_{\ell=0}^{n}(|\beta_\ell| + |\gamma_\ell| + |\delta_\ell|)\right)\right\}|q|^n. \end{aligned}$$

So if $|q| > 1 + \frac{2}{\alpha_n}\sum_{\ell=0}^{n}(|\beta_\ell| + |\gamma_\ell| + |\delta_\ell|)$ then $|p(q) * (1-q)| > 0$ and $p(q) * (1-q) \ne 0$. Since the only zeros of $p(q) * (1-q)$ are $q = 1$ and the zeros of p, for $|q| > 1 + \frac{2}{\alpha_n}\sum_{\ell=0}^{n}(|\beta_\ell| + |\gamma_\ell| + |\delta_\ell|)$ we have $p(q) \ne 0$. That is, all the zeros of p lie in

$$|q| \le 1 + \frac{2}{\alpha_n}\sum_{\ell=0}^{n}(|\beta_\ell| + |\gamma_\ell| + |\delta_\ell|),$$

as claimed. □

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