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SOME CLASSES OF POLYNOMIALS SATISFYING SENDOV'S CONJECTURE

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Abstract

In this paper, a relationship between the zeros and critical points of a polynomial $p(z)$ is established. The relationship is used to prove Sendov's conjecture in some special cases.

1. Introduction

The conjecture of interest in this paper is known variously as the Ilieff Conjecture, the Ilieff-Sendov Conjecture, and the Sendov Conjecture. It was originally posed by Bulgarian mathematician Blagovest Sendov in 1958, but often attributed to Ilieff because of a reference in Hayman's *Research Problems in Function Theory* [6] in 1967.

THE SENDOV CONJECTURE.. *If all the zeros of a polynomial* $p(z)$ *lie in* $|z| \leq 1$ *and if* z_0 *is a zero of* $p(z)$*, then there is a zero of its derivative* $p'(z)$ *in the disk* $|z - z_0| \leq 1$.

Hundreds of papers have been published on this conjecture (for details see [7]). Though progress has been made, the general conjecture remains open. Motivated by the progress, Sheil-Small [14, p. 206] has commented:

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> "Most of the results obtained point strongly in favour of the conjecture being correct, although there are occasional hints in the opposite direction. My own view is that odds of 4–1 in favour of the conjecture are cautious odds in the circumstances."

Rahman and Schmeisser [9, p. 239] strike a more conservative tone:

> "After more than thirty years of research on Sendov's conjecture, it seems that the standard methods from the theory of polynomials have been exhausted and new approaches are needed."

In 1968, Rubenstein [10] showed that Sendov's conjecture holds for all polynomials p whose zeros lie on $|z| = 1$. In the same paper, Rubenstein also proved the conjecture for all polynomials of degree 3 and 4. In 1969 Schmeisser [11] showed that, if the convex hull containing all zeros of p has its vertices on $|z| = 1$, then p satisfies the conjecture (for the proof see [9, Theorem 7.3.4]).

Some partial results about Sendov's conjecture include:

1. The conjecture holds for polynomials with real and non-positive coefficients [12].
2. The conjecture holds for polynomials with at most six distinct zeros [2],
3. The conjecture holds for polynomials of degree less than or equal to 8 [3].
4. The conjecture holds if, instead of considering disks centered at zeros of a polynomial of radius 1, we consider such disks of radius 1.08331641 [1].

In this paper, we prove a relation between the zeros of a polynomial and its critical points and use it to obtain some results related to Sendov's conjecture.

2. A useful relationship between the Zeros and Critical points

Suppose $p(z)$ is a polynomial of degree n with zeros $z_1, z_2, \ldots, z_n$ and leading coefficient a_n so that $p(z) = a_n \prod_{k=1}^{n}(z - z_k)$. Let the zeros of p' be ζ_k for $k = 1, 2, \ldots, n-1$, so that we can write

$$p'(z) = na_n \prod_{k=1}^{n-1}(z - \zeta_k). \tag{1}$$

We have

$$p'(z) = a_n \frac{d}{dz}\left[\prod_{k=1}^{n}(z - z_k)\right] = a_n \sum_{j=1}^{n} \prod_{k=1, k\neq j}^{n} (z - z_k). \tag{2}$$

So for any j with $1 \leq j \leq n$, we have $p'(z_j) = a_n \prod_{k=1,k\neq j}^{n}(z_j - z_k)$. But from (1) $p'(z_j) = na_n \prod_{k=1}^{n-1}(z_j - \zeta_k)$ and therefore

$$na_n \prod_{k=1}^{n-1}(z_j - \zeta_k) = a_n \prod_{k=1,k\neq j}^{n}(z_j - z_k).$$

This gives

$$n \prod_{k=1}^{n-1}|z_j - \zeta_k| = \prod_{k=1,k\neq j}^{n}|z_j - z_k| \text{ for } 1 \leq j \leq n. \tag{3}$$

Hence we have the following:

LEMMA 2.1. *If $p(z)$ is a polynomial of degree n with zeros $z_1, z_2, \ldots, z_n$ and critical points $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$ then*

$$\prod_{k=1}^{n-1}|z_j - \zeta_k| = \frac{1}{n}\prod_{k=1,k\neq j}^{n}|z_j - z_k| \textit{ for } 1 \leq j \leq n.$$

REMARK. If $\prod_{k=1,k\neq j}^{n}|z_j - z_k| \leq n$, then $\prod_{k=1}^{n-1}|z_j - \zeta_k| \leq 1$ for $1 \leq j \leq n$ that is $|z_j - \zeta_k| \leq 1$ for atleast one $k, 1 \leq k \leq n-1$ and $1 \leq j \leq n$. This shows that in this case there is atleast one critical point of $p(z)$ which lies in the circle $|z - z_j| \leq 1$, for all $j, 1 \leq j \leq n$.

As an application of the above lemma we now prove some results which show that Sendov's Conjecture is true for certian classes of polynomials and accordingly we have the following:

THEOREM 2.1. *Let $p(z) = a_n \prod_{k=1}^{n}(z - z_k)$ be a polynomial of degree $n \geq 2$ with its zeros satisfying $|z_k| \leq 1$ for $k = 1, 2, \ldots, n$ and with*

$$max_{|z|=1}|p(z)| \leq |a_n|\frac{2n}{n+1}.$$

Then each of the disks $|z - z_k| \leq 1$, $k = 1, 2, \ldots, n$, must contains a zero of $p'(z)$.

It is natural to ask if there are any polynomials satisfying the conditions of Theorem 2.1. For this consider the polynomial $p_n(z) = 1 + z + z^2 + z^3 + \cdots + z^{n-1} + 2nz^n$ for which $\max_{|z|=1}|p(z)| = 3n$. By the classical Eneström-Kakeya Theorem [5] all zeros of p_n lie in $|z| \leq 1$ as required by Theorem 2.1. Also it can be easily verified

$$2n = |a_n| \geq \frac{n+1}{2n}\max_{|z|=1}|p(z)|$$

holds for $n \geq 3$. Therefore for each $n \geq 3$, polynomial p_n satisfies the hypotheses of Theorem 2.1. This shows that there exists polynomial of every degree greater than 2 to which Theorem 2.1 applies.

THEOREM 2.2. *Let $p(z) = a_n \prod_{k=1}^{n}(z - z_k)$ be a polynomial of degree $n \geq 2$ with its zeros satisfying $|z_k| \leq 1$ for $k = 1, 2, \ldots, n$. Let m be a natural number such that $2^{m-1} \leq n < 2^m$. Suppose each disk $|z - z_k| \leq 1$, for $k = 1, 2, \ldots, n$ contains at least $n - m$ zeros of p other than z_k. Then Sendov's conjecture holds true for such polynomials.*

3. Illustration

We can illustrate applications of Theorem 2.2 by finding polynomials for which the zeros are somewhat clustered together. For example, let $m \in \mathbb{N}$, $m \geq 4$, and $n = 2^{m-1}$. Define three disks, $D_- = \{z \in \mathbb{C} \mid |z + 1/2| \leq 1/4\}$, $D_+ = \{z \in \mathbb{C} \mid |z - 1/2| \leq 1/4\}$, and $D_0 = \{z \in \mathbb{C} \mid |z| \leq 1/4\}$. Let p be a polynomial of degree n with $n - 2m + 2$ of its zeros anywhere in D_0, $m - 1$ of its zeros anywhere in D_-, and $m - 1$ of its zeros anywhere in D_+. Notice that disks D_- and D_0 are contained in $|z + 1/4| = 1/2$, and disks D_0 and D_+ are contained in $|z - 1/4| = 1/2$; (see Figure 1). So any closed unit disk centered at a zero in D_- or centered at a zero in D_+ must contain at least $m - n$ other zeros of p. Of course, a closed unit disk centered at a zero in D_0 must contain all zeros of p. So polynomial p satisfies Sendov's conjecture by Theorem 2.2. One sees how this approach can be used to generate other polynomials satisfying the hypotheses of Theorem 2.2.

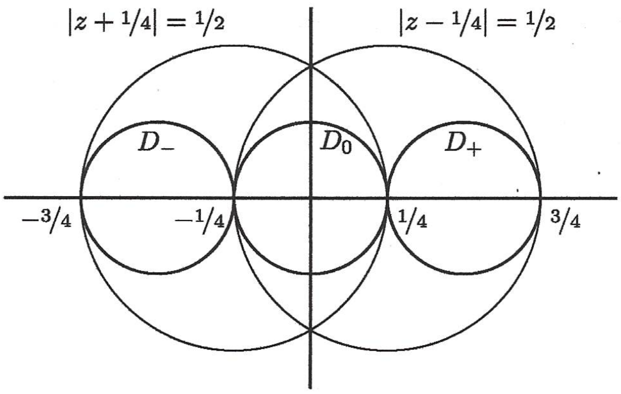

Fig. 1. Locations of zeros of a polynomial to which Theorem 2.2 applies.

We next apply the Lemma 2.1 to establish that a certain zero of a particular form of polynomial contains within distance 1 some critical point of the polynomial.

THEOREM 3.1. *Let* $p(z) = a_n(z - \beta)q(z)$ *be a polynomial of degree* n *where* $|\beta| \leq 1$. *Suppose that all the zeros of* q *lie in* $|z| \leq 1$, *and* $\max_{|z|=1} |q(z)| \leq n$. *Then the disk* $|z - \beta| \leq 1$ *contains a critical point of* p.

Notice that Theorem 3.1 applies when $q(z) = z^{n-1} + z^{n-2} + \cdots + z + 1$. By a result of Schmeisser [11] on the convex hull containing the zeros of p, the conclusion of Theorem 3.1 holds if β is in the convex hull determined by the zeros of this particular q. Theorem 3.1 therefore contributes a new result when $|\beta| < 1$ and β is outside the convex hull determined by the $n - 1$ zeros of $q(z) = z^{n-1} + z^{n-2} + \cdots + z + 1$.

Concerning the Sendov's conjecture Rubenstein [10, Theorem 3] proved the following result.

THEOREM 3.2. *If* $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1z + a_0$ *with* $p(z_1) = 0$ *and* $|p'(z_1)| < n$, *then* p' *has a zero in* $|z - z_1| < 1$.

Here, as an application of Lemma 2.1 we give an easy proof of this theorem.

For the proof of these results, we also need the following lemma due to Donaldson and Rahman [4].

LEMMA 3.1. *If* p *is a polynomial of degree* n, $p(\beta) = 0$, *and* $\max_{|z|=1} |p(z)| \leq 1$, *then*

$$\max_{|z|=1} \left| \frac{p(z)}{z - \beta} \right| \leq \frac{n+1}{2}.$$

4. Proofs of the Theorems

PROOF OF THEOREM 2.1.. Assume there is some z_k, say $z_1$violating the claim. Then z_1 must be a zero of p of multiplicity 1, or else z_1 would also be a zero of p'. Since $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$ are the zeros of p', therefore $|z_1 - \zeta_k| > 1$ for $k = 1, 2, \ldots, n - 1$. Let $M = \max_{|z|=1} |p(z)|$, so that $\max_{|z|=1} |p(z)/M| = 1$. Then by (3)

$$\prod_{k=2}^{n}(z_1 - z_k) = n \prod_{k=1}^{n-1}(z_1 - \zeta_k). \tag{4}$$

Applying Lemma 3.1 to $p(z)/M$ with $\beta = z_1$, we get

$$\max_{|z|=1} \left| \frac{p(z)/M}{z - z_1} \right| \leq \frac{n+1}{2}. \tag{5}$$

Now

$$\frac{p(z)}{z-z_1} = \frac{a_n \prod_{k=1}^{n}(z-z_k)}{z-z_1} = a_n \prod_{k=2}^{n}(z-z_k).$$

Therefore from equation (5) we have

$$\frac{|a_n|}{M} \prod_{k=2}^{n} |z-z_k| \leq \frac{n+1}{2}.$$

Since $\frac{a_n}{M} \prod_{k=2}^{n}(z-z_k)$ is analytic in $\mathbb{C}$ and $|z_1| \leq 1$ by hypothesis, Therefore by the Maximimum Modulus principle,

$$\frac{|a_n|}{M} \prod_{k=2}^{n} |z_1-z_k| \leq \frac{n+1}{2}.$$

Hence by equation (4) we have

$$\frac{n|a_n|}{M} \prod_{k=1}^{n-1} |z_1-\zeta_k| \leq \frac{n+1}{2}. \tag{6}$$

Since $|z_1-\zeta_k| > 1$ for $k = 1, 2, \ldots, n-1$ by assumption, we have, $\frac{n|a_n|}{M} < \frac{n|a_n|}{M} \prod_{k=1}^{n-1} |z_1-\zeta_k|$ and so from inequality (6), $\frac{n|a_n|}{M} < \frac{n+1}{2}$. which gives $|a_n| < \frac{n+1}{2n} M = \frac{n+1}{2n} \max_{|z|=1} |p(z)|$, contradicting the hypotheses. So the assumption is false and hence $|z_j - \zeta_k| \leq 1$ for $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n-1$, as claimed. □

Proof of Theorem 2.2.. Assume, to the contrary and let $|z_1 - \zeta_k| > 1$ for each $k = 1, 2, \ldots, n-1$, where $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$ are the zeros of p'. Therefore from (3),

$$n \prod_{k=1}^{n-1} |z_1-\zeta_k| = \prod_{k=2}^{n} |z_1-z_k|.$$

So that

$$\prod_{k=2}^{n} |z_1-z_k| > n. \tag{7}$$

But for at least $n-m$ of the values of $k \in \{2,3,\ldots,n\}$ we have $|z_1 - z_k| \leq 1$ and for at most $m-1$ of the values of $k \in \{1,2,\ldots,n\}$ we have $1 < |z_1 - z_k| \leq 2$. So $\prod_{k=2}^{n} |z_1 - z_k| \leq 2^{m-1} \leq n$, but this contradicts (7). So the assumption is false and the desired result follows. □

PROOF OF THEOREM 3.1.. Let the zeros of p' be $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$. Then $p'(\beta) = na_n \prod_{k=1}^{n-1}(\beta - \zeta_k)$. But $p'(\beta) = a_n q(\beta)$ so that

$$n \geq \max_{|z|=1} |q(z)| \geq |q(\beta)| = n \prod_{k=1}^{n-1} |\beta - \zeta_k|$$

or $\prod_{k=1}^{n-1} |\beta - \zeta_k| \leq 1$. Hence, for some k, $|\beta - \zeta_k| \leq 1$ as claimed. □

PROOF OF THEOREM 3.2.. Let the zeros of p' be $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$. By (1), we have $p'(z_1) = n \prod_{k=1}^{n-1}(z_1 - \zeta_k)$, so $|p'(z_1)| = n \prod_{k=1}^{n-1} |z_1 - \zeta_k| < n$, or $\prod_{k=1}^{n-1} |z_1 - \zeta_k| < 1$. Hence $|z_1 - \zeta_k| < 1$ for some $k \in \{1,2,\ldots,n-1\}$, as claimed. □

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