Article

The Number of Zeros in a Disk of a Complex Polynomial with Coefficients Satisfying Various Monotonicity Conditions

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Abstract: Motivated by results on the location of the zeros of a complex polynomial with monotonicity conditions on the coefficients (such as the classical Eneström–Kakeya theorem and its recent generalizations), we impose similar conditions and give bounds on the number of zeros in certain regions. We do so by introducing a reversal in monotonicity conditions on the real and imaginary parts of the coefficients and also on their moduli. The conditions imposed are less restrictive than many of those in the current literature and hence apply to polynomials not covered by previous results. The results presented naturally apply to certain classes of lacunary polynomials. In particular, the results apply to certain polynomials with two gaps in their coefficients.

1. Introduction

The classical Eneström–Kakeya theorem concerns the location of the complex zeros of a real polynomial with nonnegative monotone coefficients. It was independently proved by Gustav Eneström in 1893 [1] and Sōichi Kakeya in 1912 [2].

Theorem 1 (Eneström–Kakeya theorem). *If $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of P lie in $|z| \leq 1$.*

A large body of literature on results related to the Eneström–Kakeya theorem now exists. For a survey of results up through 2014, see [3]. Inspired by the results of Aziz and Zargar [4] and Shah et al. [5], the present authors gave an Eneström–Kakeya-type result [6] for polynomials $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ such that $\alpha_\ell = \operatorname{Re}(a_\ell)$ and $\beta_\ell = \operatorname{Im}(a_\ell)$ for $0 \leq \ell \leq n$ where, for some positive numbers ρ_r and ρ_i, each at most 1; k_r and k_i, each at least 1; and p and q with $0 \leq q \leq p \leq n$, the coefficients satisfy

$$\rho_r \alpha_q \leq \alpha_{q+1} \leq \alpha_{q+2} \leq \cdots \leq \alpha_{p-1} \leq k_r \alpha_p$$

and

$$\rho_i \beta_q \leq \beta_{q+1} \leq \beta_{q+2} \leq \cdots \leq \beta_{p-1} \leq k_i \beta_p.$$

For such polynomials, all the zeros of P lie in the closed annulus:

$$\min \left\{ 1, |a_0| \Big/ \Big(M_q + (1 - \rho_r)|\alpha_q| - \rho_r \alpha_q + (1 - \rho_i)|\beta_q| - \rho_i \beta_q + k_r \alpha_p + (k_r - 1)|\alpha_p| \right.$$

$$\left. + k_i \beta_p + (k_i - 1)|\beta_p| + M_p + |a_n| \Big) \right\} \leq |z| \leq \Big(|a_0| + M_q + (1 - \rho_r)|\alpha_q| - \rho_r \alpha_q$$

$$+(1-\rho_i)|\beta_q| - \rho_i\beta_q + (k_r - 1)|\alpha_p| + k_r\alpha_p + (k_i - 1)|\beta_p| + k_i\beta_p + M_p\Big)\Big/|a_n|,$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. The present authors recently generalized this result [7] by adding the parameter j such that $q < j < p$ (which allows a reversal in the monotonicity condition) and using a total of six positive parameters, ρ_r, ρ'_r, ρ_i, and ρ'_i, each at most 1, and k_r and k_i, each at least 1, to consider polynomials with complex coefficients satisfying

$$\rho_r\alpha_q \le \alpha_{q+1} \le \alpha_{q+2} \le \cdots \alpha_{j-1} \le k_r\alpha_j \ge \alpha_{j+1} \ge \cdots \ge \alpha_{p-1} \ge \rho'_r\alpha_p \qquad (1)$$

and

$$\rho_i\beta_q \le \beta_{q+1} \le \beta_{q+2} \le \cdots \beta_{j-1} \le k_i\beta_j \ge \beta_{j+1} \ge \cdots \ge \beta_{p-1} \ge \rho'_i\beta_p. \qquad (2)$$

Notice that with $\rho_r = k_r = \rho'_r = 1$, $q = 0$, $j = p = n$, $0 \le a_0$, and each $\beta_\ell = 0$, the above condition implies the hypotheses of the Eneström–Kakeya theorem.

The first result concerning the number of zeros in a disk relevant to the current work was published by Mohammad in 1965. It considers polynomials with real coefficients tha satisfy the monotonicity condition of the Eneström–Kakeya theorem (with the added condition that the constant term is nonzero) and is as follows [8]:

Theorem 2. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with real coefficients such that $0 < a_0 \le a_1 \le \cdots \le a_n$. The number of zeros of $P(z)$ in the disk $|z| \le 1/2$ does not exceed $1 + (1/\log 2)\log(a_n/a_0)$.*

Another relevant result is due to Dewan [9] and concerns a monotonicity condition on the moduli of coefficients, as follows:

Theorem 3. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_\ell - \beta| \le \alpha \le \pi/2$ for $\ell = 0, 1, \ldots, n$ and $0 < |a_0| \le |a_1| \le \cdots \le |a_n|$. The number of zeros of $P(z)$ in $|z| \le 1/2$ does not exceed*

$$\frac{1}{\log 2}\log\left(\frac{|a_n|(1 + \cos\alpha + \sin\alpha) + 2\sin\alpha\sum_{\ell=0}^{n-1}|a_\ell|}{|a_0|}\right).$$

Though both Theorems 2 and 3 concern zeros in $|z| \le 1/2$, more general results exist. For example, Pukhta [10] gave the following generalization of Theorem 3, which reduces to Theorem 3 when $\delta = 1/2$:

Theorem 4. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_\ell - \beta| \le \alpha \le \pi/2$ for $\ell = 0, 1, \ldots, n$ and $0 < |a_0| \le |a_1| \le \cdots \le |a_n|$. For $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \le \delta$ does not exceed*

$$\frac{1}{\log 1/\delta}\log\left(\frac{|a_n|(1 + \cos\alpha + \sin\alpha) + 2\sin\alpha\sum_{\ell=0}^{n-1}|a_\ell|}{|a_0|}\right).$$

Recently, the number of zeros in a disk of a polynomial with coefficients satisfying a monotonicity condition, but with extra multiplicative terms on some of the coefficients, has been presented. Rather et al. [11], for example, considered polynomials with real coefficients satisfying

$$a_0 \le a_1 \le \cdots \le a_{n-r-1} \le k_r a_{n-r} \le k_{r-1} a_{n-r+1} \le \cdots \le k_1 a_{n-1} \le k_0 a_n$$

for some $k_\ell \ge 1$, $\ell = 0, 1, \ldots, r$ where $0 \le r \le n - 1$. Rather et al. [12] (in a publication different from the previously cited one) similarly considered a monotonicity condition

but with extra additive terms on some of the coefficients. For example, they considered polynomials with real coefficients satisfying

$$a_0 \leq a_1 \leq \cdots \leq a_{n-r-1} \leq k_r + a_{n-r} \leq k_{r-1} + a_{n-r+1} \leq \cdots \leq k_1 + a_{n-1} \leq k_0 + a_n$$

for some $k_\ell \geq 0$, $\ell = 0, 1, \ldots, r$ where $0 \leq r \leq n - 1$. These results of Rather et al. generalize and refine the earlier results.

The purpose of this paper is to consider complex polynomials satisfying Conditions (1) and (2) (and a related condition on the moduli of the coefficients) and to give results concerning the number of zeros in a disk.

2. Results

For a polynomial of degree n with complex coefficients a_ℓ, $0 \leq \ell \leq n$, where $\alpha_\ell = \text{Re}(a_\ell)$ and $\beta_\ell = \text{Im}(a_\ell)$, we impose the conditions of Equations (1) and (2) to obtain the following.

Theorem 5. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a complex polynomial of degree n with complex coefficients where $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$ which satisfies, for some real $\rho_r, \rho_r', \rho_i, \rho_i', k_r$, and k_i where $0 < \rho_r \leq 1, 0 < \rho_r' \leq 1, 0 < \rho_i \leq 1, 0 < \rho_i' \leq 1, k_r \geq 1$, and $k_i \geq 1$, the conditions*

$$\rho_r \alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{j-1} \leq k_r \alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_{p-1} \geq \rho_r' \alpha_p,$$

$$\rho_i \beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{j-1} \leq k_i \beta_j \geq \beta_{j+1} \geq \cdots \geq \beta_{p-1} \geq \rho_i' \beta_p$$

where $0 \leq q \leq p \leq n$. The number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is then less than $(1/\log(1/\delta)) \log(M/|a_0|)$ for $0 < \delta < 1$, where

$$
\begin{aligned}
M &= |a_0| + M_q - \rho_r \alpha_q + |\alpha_q|(1 - \rho_r) + 2|\alpha_j|(k_r - 1) + 2k_r \alpha_j + |\alpha_p|(1 - \rho_r') \\
&\quad - \rho_r' \alpha_p - \rho_i \beta_q + |\beta_q|(1 - \rho_i) + 2|\beta_j|(k_i - 1) + 2k_i \beta_j \\
&\quad + |\beta_p|(1 - \rho_i') - \rho_i' \beta_p + M_p + |a_n|,
\end{aligned}
$$

$M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$, and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

We now consider a condition similar to that given in Equations (1) and (2) but imposed on the moduli of the complex coefficients instead of on the real and imaginary parts.

Theorem 6. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients satisfying $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2$, $\ell = q, q+1, \ldots, p$ such that for real k, ρ, ρ' where $k \geq 1, 0 < \rho \leq 1$, $0 < \rho' \leq 1$, we have*

$$\rho|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{j-1}| \leq k|a_j| \geq |a_{j+1}| \geq \cdots \geq |a_{p-1}| \geq \rho'|a_p|,$$

where $0 \leq q \leq p \leq n$. The number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is then less than $(1/\log(1/\delta)) \log(M/|a_0|)$ for $0 < \delta < 1$, where

$$
\begin{aligned}
M &= |a_0| + M_q + |a_q| + \rho|a_q|(\sin \alpha - \cos \alpha - 1) + 2 \sum_{\ell=q+1}^{j-1} |a_\ell| \sin \alpha - 2|a_j| \\
&\quad + 2k|a_j|(\cos \alpha + \sin \alpha + 1) + 2 \sum_{\ell=j+1}^{p-1} |a_\ell| \sin \alpha + |a_p| \\
&\quad + \rho'|a_p|(\sin \alpha - \cos \alpha - 1) + M_p + |a_n|,
\end{aligned}
$$

$M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$, and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

The class of lacunary polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$ was introduced by Chan and Malik in 1983 [13] in connection with Bernstein's Inequality [13]. For a survey of such results, see Subsections 4.1.4, 6.4.2, and 6.4.3 of [14]. Theorems 5 and 6 naturally apply to such polynomials that satisfy the monotonicity condition on the remaining coefficients. For example, with coefficients $a_1 = a_2 = \cdots = a_{q-1} = 0$ in polynomial P, we obtain the following corollary.

Corollary 1. *Let* $P(z) = a_0 + \sum_{\ell=q}^{n} a_\ell z^\ell$ *be a complex polynomial of degree n with complex coefficients where* $\alpha_\ell = Re(a_\ell)$ *and* $\beta_\ell = Im(a_\ell)$ *which satisfies, for some real* $\rho_r, \rho'_r, \rho_i, \rho'_i, k_r,$ *and* k_i *where* $0 < \rho_r \le 1, 0 < \rho'_r \le 1, 0 < \rho_i \le 1, 0 < \rho'_i \le 1, k_r \ge 1,$ *and* $k_i \ge 1,$ *the conditions*

$$\rho_r \alpha_q \le \alpha_{q+1} \le \cdots \le \alpha_{j-1} \le k_r \alpha_j \ge \alpha_{j+1} \ge \cdots \ge \alpha_{p-1} \ge \rho'_r \alpha_p,$$

$$\rho_i \beta_q \le \beta_{q+1} \le \cdots \le \beta_{j-1} \le k_i \beta_j \ge \beta_{j+1} \ge \cdots \ge \beta_{p-1} \ge \rho'_i \beta_p$$

where $0 \le q \le p \le n$. *The number of zeros of $P(z)$ in the disk $|z| \le \delta$ is then less than* $(1/\log(1/\delta)) \log(M/|a_0|),$ *where M is as given in Theorem 5, $M_q = 0$, and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|.$*

A similar corollary follows from Theorem 6. In addition, Theorems 5 and 6 naturally apply to lacunary polynomials with two gaps in their coefficients. For example, with the coefficients $a_1 = a_2 = \cdots = a_{q-1} = 0$ and $a_{p+1} = a_{p+2} = \cdots = a_{n-1} = 0$ in polynomial P, we obtain the following corollary.

Corollary 2. *Let* $P(z) = a_0 + \sum_{\ell=q}^{p} a_\ell z^\ell + a_n$ *be a complex polynomial of degree n with complex coefficients where* $\alpha_\ell = Re(a_\ell)$ *and* $\beta_\ell = Im(a_\ell)$ *which satisfies, for some real* $\rho_r, \rho'_r, \rho_i, \rho'_i, k_r,$ *and* k_i *where* $0 < \rho_r \le 1, 0 < \rho'_r \le 1, 0 < \rho_i \le 1, 0 < \rho'_i \le 1, k_r \ge 1,$ *and* $k_i \ge 1,$ *the conditions*

$$\rho_r \alpha_q \le \alpha_{q+1} \le \cdots \le \alpha_{j-1} \le k_r \alpha_j \ge \alpha_{j+1} \ge \cdots \ge \alpha_{p-1} \ge \rho'_r \alpha_p,$$

$$\rho_i \beta_q \le \beta_{q+1} \le \cdots \le \beta_{j-1} \le k_i \beta_j \ge \beta_{j+1} \ge \cdots \ge \beta_{p-1} \ge \rho'_i \beta_p$$

where $0 \le q \le p \le n$. *The number of zeros of $P(z)$ in the disk $|z| \le \delta$ is then less than* $(1/\log(1/\delta)) \log(M/|a_0|),$ *where M is as given in Theorem 5 and $M_q = M_p = 0$.*

A similar corollary follows from Theorem 6.

The introduction of the reversal of the inequality at index j allows us to shift the point at which the reversal occurs. This flexibility allows us to apply Theorems 5 and 6 to a larger collection of polynomials than some of the other current results in the literature on this topic.

3. Lemmas

The results regarding the number of zeros we consider are all based on the following theorem, which appears in Titchmarsh's *The Theory of Functions* ([15], page 280).

Lemma 1. *Let $F(z)$ be analytic in $|z| \le R$. Let $|F(z)| \le M$ in the disk $|z| \le R$ and suppose $F(0) \ne 0$. For $0 < \delta < 1$, the number of zeros of F in the disk $|z| \le \delta R$ does not exceed* $(1/\log(1/\delta)) \log(M/|F(0)|).$

The following lemma is due to Govil and Rahman [16].

Lemma 2. *Let $z, z' \in \mathbb{C}$ with $|z| \ge |z'|$. Suppose that $|\arg z^* - \beta| \le \alpha \le \pi/2$ for $z^* \in \{z, z'\}$ and for some real α and β. Then,*

$$|z - z'| \le (|z| - |z'|) \cos \alpha + (|z| + |z'|) \sin \alpha.$$

4. Proofs of the Results

Proof of Theorem 5. Consider

$$F(z) = (1 - z)P(z) = z_0 + \sum_{\ell=1}^{n}(a_\ell - a_{\ell-1})z^\ell - a_n z^{n+1}.$$

For $|z| = 1$, we have

$$
\begin{aligned}
|F(z)| &\leq |a_0| + \sum_{\ell=1}^{n}|a_\ell - a_{\ell-1}||z|^\ell + |a_n||z|^{n+1} = |a_0| + \sum_{\ell=1}^{n}|a_\ell - a_{\ell-1}| + |a_n| \\
&= |a_0| + \sum_{\ell=1}^{q}|a_\ell - a_{\ell-1}| + \sum_{\ell=q+1}^{p}|\alpha_\ell + i\beta_\ell - \alpha_{\ell-1} - i\beta_{\ell-1}| \\
&\quad + \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}| + |a_n| \\
&\leq |a_0| + M_q + \sum_{\ell=q+1}^{p}|\alpha_\ell - \alpha_{\ell-1}| + \sum_{\ell=q+1}^{p}|\beta_\ell - \beta_{\ell-1}| + M_p + |a_n|,
\end{aligned}
$$

where $M_q = \sum_{\ell=1}^{q}|a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}|$. For $|z| = 1$, we have

$$
\begin{aligned}
|F(z)| &\leq |a_0| + M_q + |\alpha_{q+1} - \rho_r\alpha_q + \rho_r\alpha_q - \alpha_q| + \sum_{\ell=q+2}^{j-1}|\alpha_\ell - \alpha_{\ell-1}| \\
&\quad + |\alpha_j - k_r\alpha_j + k_r\alpha_j - \alpha_{j-1}| + |\alpha_{j+1} - k_r\alpha_j + k_r\alpha_j - \alpha_j| + \sum_{\ell=j+2}^{p-1}|\alpha_\ell - \alpha_{\ell-1}| \\
&\quad + |\alpha_p - \rho_r'\alpha_p + \rho_r'\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_i\beta_q + \rho_i\beta_q - \beta_q| \\
&\quad + \sum_{\ell=q+2}^{j-1}|\beta_\ell - \beta_{\ell-1}| + |\beta_j - k_i\beta_j + k_i\beta_j - \beta_{j-1}| + |\beta_{j+1} - k_i\beta_j + k_i\beta_j - \beta_j| \\
&\quad + \sum_{\ell=j+2}^{p-1}|\beta_\ell - \beta_{\ell-1}| + |\beta_p - \rho_i'\beta_p + \rho_i'\beta_p - \beta_{p-1}| + M_p + |a_n| \\
&\leq |a_0| + M_q + |\alpha_{q+1} - \rho_r\alpha_q| + |\rho_r\alpha_q - \alpha_q| + \sum_{\ell=q+2}^{j-1}|\alpha_\ell - \alpha_{\ell-1}| + |\alpha_j - k_r\alpha_j| \\
&\quad + |k_r\alpha_j - \alpha_{j-1}| + |\alpha_{j+1} - k_r\alpha_j| + |k_r\alpha_j - \alpha_j| + \sum_{\ell=j+2}^{p-1}|\alpha_\ell - \alpha_{\ell-1}| + |\alpha_p - \rho_r'\alpha_p| \\
&\quad + |\rho_r'\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_i\beta_q| + |\rho_i\beta_q - \beta_q| + \sum_{\ell=q+2}^{j-1}|\beta_\ell - \beta_{\ell-1}| \\
&\quad + |\beta_j - k_i\beta_j| + |k_i\beta_j - \beta_{j-1}| + |\beta_{j+1} - k_i\beta_j| + |k_i\beta_j - \beta_j| + \sum_{\ell=j+2}^{p-1}|\beta_\ell - \beta_{\ell-1}| \\
&\quad + |\beta_p - \rho_i'\beta_p| + |\rho_i'\beta_p - \beta_{p-1}| + M_p + |a_n| \\
&= |a_0| + M_q + (\alpha_{q+1} - \rho_r\alpha_q) + |\alpha_q|(1 - \rho_r) + \sum_{\ell=q+2}^{j-1}(\alpha_\ell - \alpha_{\ell-1}) + |\alpha_j|(k_r - 1) \\
&\quad + (k_r\alpha_j - \alpha_{j-1}) + (k_r\alpha_j - \alpha_{j+1}) + |\alpha_j|(k_r - 1) + \sum_{\ell=j+2}^{p-1}(\alpha_{\ell-1} - \alpha_\ell) \\
&\quad + |\alpha_p|(1 - \rho_r') + (\alpha_{p-1} - \rho_r'\alpha_p) + (\beta_{q+1} - \rho_i\beta_q) + |\beta_q|(1 - \rho_i)
\end{aligned}
\tag{3}
$$

$$+ \sum_{\ell=q+2}^{j-1} (\beta_\ell - \beta_{\ell-1}) + |\beta_j|(k_i - 1) + (k_i\beta_j - \beta_{j-1}) + (k_i\beta_j - \beta_{j+1})$$

$$+ |\beta_j|(k_i - 1) + \sum_{\ell=j+2}^{p-1} (\beta_{\ell-1} - \beta_\ell) + |\beta_p|(1 - \rho_i') + (\beta_{p-1} - \rho_i'\beta_p) + M_p + |a_n|$$

$$= |a_0| + M_q - \rho_r\alpha_q + |\alpha_q|(1 - \rho_r) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j + |\alpha_p|(1 - \rho_r')$$
$$- \rho_r'\alpha_p - \rho_i\beta_q + |\beta_q|(1 - \rho_i) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j$$
$$+ |\beta_p|(1 - \rho_i') - \rho_i'\beta_p + M_p + |a_n|.$$

Since $F(z)$ is analytic in $|z| \leq 1$, by Lemma 1 and the maximum modulus theorem, the number of zeros of $F(z)$ (and hence of $P(z)$) in $|z| \leq \delta$ is less than or equal to $(1/\log(1/\delta))\log(M/|a_0|)$, where $0 < \delta < 1$, as claimed. $\square$

Proof of Theorem 6. Consider $F(z) = (1 - z)P(z)$. For $|z| = 1$, we have

$$
\begin{aligned}
|F(z)| \leq\ & |a_0| + M_q + |a_{q+1} - \rho a_q| + |\rho_1 a_q - a_q| + \sum_{\ell=q+2}^{j-1} |a_\ell - a_{\ell-1}| \\
& + |a_j - ka_j| + |ka_j - a_{j-1}| + |a_{j+1} - ka_j| + |ka_j - a_j| + \sum_{\ell=j+2}^{p-1} |a_\ell - a_{\ell-1}| \\
& + |a_p - \rho'a_p| + |\rho'a_p - a_{p-1}| + M_p + |a_n| \text{ as in (3)} \\
\leq\ & |a_0| + M_q + |a_{q+1}|\cos\alpha - \rho|a_q|\cos\alpha + |a_{q+1}|\sin\alpha + \rho|a_q|\sin\alpha \\
& + |a_q|(1 - \rho) + \sum_{\ell=q+2}^{j-1} |a_\ell|\cos\alpha - \sum_{\ell=q+2}^{j-1} |a_{\ell-1}|\cos\alpha + \sum_{\ell=q+2}^{j-1} |a_\ell|\sin\alpha \\
& + \sum_{\ell=q+2}^{j-1} |a_{\ell-1}|\sin\alpha + |a_j|(k - 1) + k|a_j|\cos\alpha - |a_{j-1}|\cos\alpha + k|a_j|\sin\alpha \\
& + |a_{j-1}|\sin\alpha + k|a_j|\cos\alpha - |a_{j+1}|\cos\alpha + k|a_j|\sin\alpha + |a_{j+1}|\sin\alpha \\
& + |a_j|(k - 1) + \sum_{\ell=j+2}^{p-1} |a_{\ell-1}|\cos\alpha - \sum_{\ell=j+2}^{p-1} |a_\ell|\cos\alpha + \sum_{\ell=j+2}^{p-1} |a_{\ell-1}|\sin\alpha \\
& + \sum_{\ell=j+2}^{p-1} |a_\ell|\sin\alpha + |a_p|(1 - \rho') + |a_{p-1}|\cos\alpha - \rho'|a_p|\cos\alpha + |a_{p-1}|\sin\alpha \\
& + \rho'|a_p|\sin\alpha + M_p + |a_n| \text{ by Lemma 2.}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|F(z)| \leq\ & |a_0| + M_q + |a_{q+1}|\cos\alpha - \rho|a_q|\cos\alpha + |a_{q+1}|\sin\alpha + \rho|a_q|\sin\alpha \\
& + |a_q|(1 - \rho) + |a_{j-1}|\cos\alpha + \sum_{\ell=q+2}^{j-2} |a_\ell|\cos\alpha - |a_{q+1}|\cos\alpha \\
& - \sum_{\ell=q+2}^{j-2} |a_\ell|\cos\alpha + |a_{j-1}|\sin\alpha + \sum_{\ell=q+2}^{j-2} |a_\ell|\sin\alpha + |a_{q+1}|\sin\alpha \\
& + \sum_{\ell=q+2}^{j-2} |a_\ell|\sin\alpha + |a_j|(k - 1) + k|a_j|\cos\alpha - |a_{j-1}|\cos\alpha + k|a_j|\sin\alpha \\
& + |a_{j-1}|\sin\alpha + k|a_j|\cos\alpha - |a_{j+1}|\cos\alpha + k|a_j|\sin\alpha + |a_{j+1}|\sin\alpha \\
& + |a_j|(k - 1) + |a_{j+1}|\cos\alpha + \sum_{\ell=j+2}^{p-2} |a_\ell|\cos\alpha - |a_{p-1}|\cos\alpha
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\ell=j+2}^{p-2} |a_\ell| \cos\alpha + |a_{j+1}| \sin\alpha + \sum_{\ell=j+2}^{p-2} |a_\ell| \sin\alpha + |a_{p-1}| \sin\alpha \\
&+ \sum_{\ell=j+2}^{p-2} |a_\ell| \sin\alpha + |a_p|(1-\rho') + |a_{p-1}| \cos\alpha - \rho'|a_p| \cos\alpha + |a_{p-1}| \sin\alpha \\
&+ \rho'|a_p| \sin\alpha + M_p + |a_n| \\
=\ & |a_0| + M_q + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2|a_{q+1}| \sin\alpha + |a_q| \\
&+ 2\sum_{\ell=q+2}^{j-2} |a_\ell| \sin\alpha - 2|a_j| + 2k|a_j|(\cos\alpha + \sin\alpha + 1) + 2|a_{j-1}| \sin\alpha \\
&+ 2|a_{j+1}| \sin\alpha + 2\sum_{\ell=j+2}^{p-2} |a_\ell| \sin\alpha + 2|a_{p-1}| \sin\alpha + |a_p| \\
&+ \rho'|a_p|(\sin\alpha - \cos\alpha - 1) + M_p + |a_n| \\
=\ & |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1} |a_\ell| \sin\alpha - 2|a_j| \\
&+ 2k|a_j|(\cos\alpha + \sin\alpha + 1) + 2\sum_{\ell=j+1}^{p-1} |a_\ell| \sin\alpha + |a_p| \\
&+ \rho'|a_p|(\sin\alpha - \cos\alpha - 1) + M_p + |a_n|.
\end{aligned}
$$

Since $F(z)$ is analytic in $|z| \leq 1$, by Lemma 1 and the maximum modulus theorem, the number of zeros of $F(z)$ (and hence of $P(z)$) in $|z| \leq \delta$ is less than or equal to $(1/\log(1/\delta)) \log(M/|a_0|)$, where $0 < \delta < 1$, as claimed. $\square$

5. Discussion

As explained in the Introduction, the hypotheses applied in this paper build on similar hypotheses in the setting of results on the the location of zeros of a complex polynomial, namely, the Eneström–Kakeya theorem and its generalizations.

Applications related to the zeros of polynomials run from the graphing of polynomials using the zeros of the first and second derivatives to the computation of eigenvalues from the characteristic polynomial of a (real or complex) matrix. For example, equilibria of systems of differential equations and the stability classification of equilibria are based on complex eigenvalues. A number of real-world models involve transition matrices, which must be diagonalized, and this process involves the zeros of the characteristic polynomial. In graph theory, a graph has adjacency and chromatic polynomials, the zeros of which give information about the graph itself. The results of the paper have potential applications in each of these settings.

Future research could involve loosening or revising the monotonicity conditions of Theorems 5 and 6. For example, the monotonicity conditions of Rather et al. in [11,12], mentioned in the Introduction, could be imposed on the real and complex parts of the coefficients and on the moduli of the coefficients to produce related results. Theorems 5 and 6 concern a single reversal in the monotonicity condition, so this could be generalized to multiple reversals. In addition, combinations of the monotonicity conditions presented here could be combined with others in the literature (such as those in [11,12]).

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