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An L^p Inequality for a Polynomial and Its Derivative

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Let $P(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu)$, $a_n \neq 0$ be a polynomial of degree n. It is known that if $|z_\nu| \geq K_\nu \geq 1$, $1 \leq \nu \leq n$, then for $p \geq 1$,

$$\left(\int_0^{2\pi} |P'(e^{i\theta})|^p \, d\theta\right)^{1/p} \leq nF_p \left(\int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta\right)^{1/p},$$

where

$$F_p = \left\{2\pi \Big/ \int_0^{2\pi} |t_0 + e^{i\theta}|^p \, d\theta\right\}^{1/p},$$

and

$$t_0 = \begin{cases} 1 + n \Big/ \sum_{\nu=1}^{n} \dfrac{1}{(K_\nu - 1)} & \text{if } K_\nu > 1 \text{ for all } \nu, 1 \leq \nu \leq n \\ 1 & \text{if } K_\nu = 1 \text{ for some } \nu, 1 \leq \nu \leq n. \end{cases}$$

This inequality is best possible in the case $K_\nu = 1$, $1 \leq \nu \leq n$, and equality holds for the polynomial $(z + 1)^n$. In this paper, we extend the above inequality to values of $p \in [0, 1)$ and thus conclude that this inequality in fact holds for all $p \geq 0$. © 1995 Academic Press, Inc.

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1. Introduction and Statement of Results

Let $\mathcal{P}_n$ be the set of all polynomials (over the complex field) of degree less than or equal to n. For $P \in \mathcal{P}_n$, define

$$\|P\|_\infty = \max_{|z|=1} |P(z)|,$$

$$\|P\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta\right)^{1/p} \qquad \text{for } 0 < p < \infty,$$

and

$$\|P\|_0 = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log |P(e^{i\theta})| \, d\theta\right).$$

A classical result of Bernstein [2] states that if $P \in \mathcal{P}_n$ then

$$\|P'\|_\infty \leq n\|P\|_\infty. \tag{1.1}$$

Also it is known (see Zygmund [9] and Arestov [1]) that if $P \in \mathcal{P}_n$, then

$$\|P'\|_p \leq n\|P\|_p, \qquad 0 \leq p \leq \infty \tag{1.2}$$

with equality if and only if $P(z) = \lambda z^n$, $\lambda \in \mathbf{C}$.

If a restriction is put on the location of the zeros of P, then the bound in (1.1) and (1.2) can be improved. If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < 1$ then (see Lax [7] and DeBruijn [3])

$$\|P'\|_\infty \leq \frac{n}{2}\|P\|_\infty \tag{1.3}$$

and

$$\|P'\|_p \leq \frac{n\|P\|_p}{\|1 + z^n\|_p}, \qquad 1 \leq p \leq \infty. \tag{1.4}$$

Rahman and Schmeisser [8] proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < 1$ then (1.4) holds for $p \in [0, 1)$ as well. This result is best possible with equality holding for $P(z) = (1 + z)^n$.

As a generalization of an inequality due to Govil and Labelle [5] and of inequality (1.4) due to DeBruijn [3], Gardner and Govil [4] proved

THEOREM A. *If* $P(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu)$, $a_n \neq 0$ *is a polynomial of degree* n, $|z_\nu| \geq K_\nu \geq 1$, $1 \leq \nu \leq n$, *then*

$$\|P'\|_p \leq \frac{n}{\|t_0 + z\|_p} \|P\|_p, \qquad 1 \leq p \leq \infty, \tag{1.5}$$

where

$$t_0 = \begin{cases} 1 + n \Big/ \sum_{\nu=1}^{n} \dfrac{1}{(K_\nu - 1)} & \text{if } K_\nu > 1 \text{ for all } \nu, 1 \leq \nu \leq n \\ 1 & \text{if } K_\nu = 1 \text{ for some } \nu, 1 \leq \nu \leq n. \end{cases}$$

The result is best possible in the case $K_\nu = 1$, $1 \leq \nu \leq n$, *and equality holds for the polynomial* $(z + 1)^n$.

The aim of this paper is to prove that the above inequality (1.5) in fact holds for $0 \leq p \leq \infty$. We prove

THEOREM. *Under the hypotheses of Theorem* A,

$$\|P'\|_p \leq \frac{n}{\|t_0 + z\|_p} \|P\|_p, \qquad 0 \leq p \leq \infty, \tag{1.6}$$

where t_0 *is as in Theorem* A. *The result is best possible in the case* $K_\nu = 1$, $1 \leq \nu \leq n$, *and equality holds for the polynomial* $(z + 1)^n$.

2. LEMMAS

The following lemmas will be needed.

LEMMA 1. *If* $P \in \mathcal{P}_n$ *and* $P(z) \neq 0$ *in a closed or open circular region* D, *then*

$$nP(z) - (z - \zeta)P'(z) \neq 0 \tag{2.1}$$

for $z \in D$ *and* $\zeta \in D$. *Here, by a closed circular region, we mean the closed interior (or exterior) of a circle or a closed half-plane. An open circular region is analogously defined.*

The above lemma is due to Laguerre [6].

For $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \mathbf{C}^{n+1}$ and

$$P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \in \mathcal{P}_n,$$

define

$$\Lambda_\gamma P(z) = \sum_{\nu=0}^{n} \gamma_\nu a_\nu z^\nu.$$

The operator Λ_γ is said to be *admissible* if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbf{C}: |z| \le 1\}$,
(ii) $P(z)$ has all its zeros in $\{z \in \mathbf{C}: |z| \ge 1\}$.

LEMMA 2 [1, Theorem 4]. *Let $\Phi(x) = \Psi(\log x)$, where Ψ is a convex nondecreasing function on $\mathbf{R}$. Then for all $P \in \mathcal{P}_n$ and each admissible operator Λ_γ,*

$$\int_0^{2\pi} \Phi(|\Lambda_\gamma P(e^{i\theta})|)\, d\theta \le \int_0^{2\pi} \Phi(c(\gamma, n)|P(e^{i\theta})|)\, d\theta, \tag{2.2}$$

where $c(\gamma, n) = \max(|\gamma_0|, |\gamma_n|)$.

In particular, Lemma 2 applies with $\Phi: x \to x^p$ for every $p \in (0, \infty)$ and with $\Phi: x \to \log x$ as well. Therefore, we have

$$\|\Lambda_\gamma P\|_p \le c(\gamma, n)\|P\|_p, \qquad 0 \le p < \infty. \tag{2.3}$$

We also need the following result due to Gardner and Govil [4, Lemma 2].

LEMMA 3. *Let $P(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu)$, $a_n \ne 0$, be a polynomial of degree n. If $|z_\nu| \ge K_\nu \ge 1$, $1 \le \nu \le n$, and $Q(z) = z^n\{\overline{P(1/\bar{z})}\}$, then for $|z| = 1$,*

$$\left|\frac{Q'(z)}{P'(z)}\right| \ge t_0, \tag{2.4}$$

where t_0 is as defined in Theorem A. It is clear that $t_0 \ge 1$.

LEMMA 4. *Let z be complex and independent of α, where α is real. Then for $p > 0$*

$$\int_0^{2\pi} |1 + ze^{i\alpha}|^p\, d\alpha = \int_0^{2\pi} \big|e^{i\alpha} + |z|\big|^p\, d\alpha. \tag{2.5}$$

Proof. We can suppose that $z = re^{i\gamma}$ with $r > 0$ and γ real. Putting $\gamma + \alpha = \beta$, the left side of (2.5) is

$$= \int_0^{2\pi} |1 + re^{i(\gamma+\alpha)}|^p \, d\alpha = \int_\gamma^{2\pi+\gamma} |1 + re^{i\beta}|^p \, d\beta$$

$$= \int_0^{2\pi} |1 + re^{i\beta}|^p \, d\beta, \qquad \text{because } (1 + re^{i\beta}) \text{ has period } 2\pi \text{ in } \beta$$

$$= \int_0^{2\pi} |e^{-i\beta} + r|^p \, d\beta = \int_0^{2\pi} |e^{i\beta} + r|^p \, d\beta,$$

from which (2.5) follows.

LEMMA 5. *Let n be a positive integer and $0 \le p \le \infty$. Then for z complex,*

$$\|1 + z^n\|_p = \|1 + z\|_p. \tag{2.6}$$

Proof. Note that for $0 < p < \infty$, the left side of (2.6) is

$$= \left(\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{in\theta}|^p \, d\theta\right)^{1/p} = \left(\frac{1}{2n\pi}\int_0^{2n\pi} |1 + e^{i\theta}|^p \, d\theta\right)^{1/p}$$

$$= \left(\frac{1}{2n\pi}\sum_{k=0}^{n-1}\int_{2k\pi}^{2(k+1)\pi} |1 + e^{i\theta}|^p \, d\theta\right)^{1/p}$$

$$= \left(\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i\theta}|^p \, d\theta\right)^{1/p}, \qquad \text{because } (1 + e^{i\theta}) \text{ has period } 2\pi \text{ in } \theta,$$

from which (2.6) follows for $0 < p < \infty$. The case $p = \infty$ is trivial. To obtain (2.6) for $p = 0$, simply make $p \to 0+$.

3. PROOF OF THE THEOREM

Since by Lemma 5, $\|1 + z^n\|_p = \|1 + z\|_p$ for $p \ge 0$, hence if $K_\nu = 1$ for some ν, $1 \le \nu \le n$, our theorem reduces to the result of Rahman and Schmeisser [8] which holds for $p \ge 0$. Therefore for the proof of our theorem, it is sufficient to consider the case when all the zeros z_ν of $P(z)$ satisfy $|z_\nu| \ge K_\nu > 1$ for $1 \le \nu \le n$. This, in view of Lemma 1, implies that $nP(z) - (z - \zeta)P'(z) \ne 0$ for $|z| \le 1$ and $|\zeta| \le 1$. Now setting $\zeta = -ze^{-i\alpha}$, α real, one can easily verify that the operator Λ defined by

$$\Lambda P(z) = (e^{i\alpha} + 1)zP'(z) - ne^{i\alpha}P(z)$$

is admissible and so by Lemma 2,

$$\int_0^{2\pi} \left| (e^{i\alpha}+1)\frac{d}{d\theta}P(e^{i\theta}) - ine^{i\alpha}P(e^{i\theta}) \right|^p d\theta \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \tag{3.1}$$

for $p > 0$, which is equivalent to

$$\int_0^{2\pi} \left| \frac{d}{d\theta}P(e^{i\theta}) + e^{i\alpha}\left\{ \frac{d}{d\theta}P(e^{i\theta}) - inP(e^{i\theta}) \right\} \right|^p d\theta \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta$$

and which gives

$$\int_0^{2\pi}\int_0^{2\pi} \left| \frac{d}{d\theta}P(e^{i\theta}) + e^{i\alpha}\left\{ \frac{d}{d\theta}P(e^{i\theta}) - inP(e^{i\theta}) \right\} \right|^p d\theta\, d\alpha \leq 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta. \tag{3.2}$$

Note that by hypothesis $n \geq 1$, so $P(z)$ is not a constant, and thus $(d/d\theta)P(e^{i\theta}) \not\equiv 0$. Therefore the left side of (3.2) is

$$= \int_0^{2\pi} \left| \frac{d}{d\theta}P(e^{i\theta}) \right|^p \int_0^{2\pi} \left| 1 + e^{i\alpha}\left\{ \frac{(d/d\theta)P(e^{i\theta}) - inP(e^{i\theta})}{(d/d\theta)P(e^{i\theta})} \right\} \right|^p d\alpha\, d\theta$$

$$= \int_0^{2\pi} \left| \frac{d}{d\theta}P(e^{i\theta}) \right|^p \int_0^{2\pi} \left| e^{i\alpha} + \left| \frac{(d/d\theta)P(e^{i\theta}) - inP(e^{i\theta})}{(d/d\theta)P(e^{i\theta})} \right| \right|^p d\alpha\, d\theta$$

by Lemma 4

$$= \int_0^{2\pi} \left| \frac{d}{d\theta}P(e^{i\theta}) \right|^p \int_0^{2\pi} \left| e^{i\alpha} + \left| \frac{Q'(e^{i\theta})}{P'(e^{i\theta})} \right| \right|^p d\alpha\, d\theta,$$

where $Q(z)$ is as defined in Lemma 3

$$\geq \int_0^{2\pi} \left| \frac{d}{d\theta}P(e^{i\theta}) \right|^p \int_0^{2\pi} |e^{i\alpha} + t_0|^p \, d\alpha\, d\theta,$$

by Lemma 3 and the fact that $|e^{i\alpha} + r|$ is an increasing function of r for $r \geq 1$. The above inequality when combined with (3.2) gives

$$\left(\int_0^{2\pi} \left| \frac{d}{d\theta}P(e^{i\theta}) \right|^p d\theta \right)\left(\int_0^{2\pi} |e^{i\alpha} + t_0|^p \, d\alpha \right) \leq 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta,$$

from which the theorem follows for $p > 0$.

To obtain the inequality when $p = 0$, simply make $p \to 0+$.

If $K_\nu = 1$ for some ν, $1 \leq \nu \leq n$, our theorem reduces to the theorem of Rahman and Schmeisser [8] which is best possible and for which equality holds for the polynomial $P(z) = (z + 1)^n$.

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References

1. V. V. Arestov, On inequalities for trigonometric polynomials and their derivatives, *Izv. Akad. Nauk SSSR Ser. Mat.* **45** (1981), 3–22 [in Russian]; *Math. USSR-Izv.* **18** (1982), 1–17.
2. S. Bernstein, Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle, Collection Borel. Paris, 1926.
3. N. DeBruijn, Inequalities concerning polynomials in the complex domain, *Nederl. Akad. Wetensch. Proc.* **50** (1947), 1265–1272; *Indag. Math.* **9** (1947), 591–598.
4. R. B. Gardner and N. K. Govil, Inequalities concerning the L^p norm of a polynomial and its derivative, *J. Math. Anal. Appl.* **179** (1993), 208–213.
5. N. K. Govil and G. Labelle, On Bernstein's inequality, *J. Math. Anal. Appl.* **126** (1987), 494–500.
6. E. Laguerre, "Oeuvres," Vol. 1; "Nouvelles Annales de Mathématiques," Vol. 17, No. 2, 1878.
7. P. D. Lax, Proof of a conjecture of P. Erdös on the derivative of a polynomial, *Bull. Amer. Math. Soc.* **50** (1944), 509–513.
8. Q. I. Rahman and G. Schmeisser, L^p inequalities for polynomials, *J. Approx. Theory* **53** (1988), 26–32.
9. A. Zygmund, A remark on conjugate series, *Proc. London Math. Soc.* (2) **34** (1932), 392–400.