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# SOME INEQUALITIES FOR MAXIMUM MODULUS OF POLYNOMIALS

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*(Dedicated to Professor R. Mohanty on his 85th birthday)*

## 1. *Introduction and statement of results*

We start by defining for a polynomial $p(z) = \sum_{v=0}^{n} a_v z^v$, the quantity

$$M(p,R) = \max_{|z|=R} |p(z)|,$$

for $R \geq 0$, and we denote $M(p,1)$ by $\|p\|$. It is clear from the Maximum Modulus Principle that $M(p,R)$ is a strictly increasing function of $R$ and is defined for $0 \leq R < \infty$. In this paper we are concerned with the growth of this function $M(p,R)$, $R \geq 0$ and will be presenting some results in this direction.

The first result in this direction is an immediate consequence of the Maximum Modulus Principle and is due to S. Bernstein (see [6] or Vol. 1, p. 137 of [5]). It states that if $p(z)$ is a polynomial of degree $n$, then for $R \geq 1$,

$$M(p,R) \leq R^n \|p\|. \tag{1.1}$$

The result is best possible and equality holds if and only if $p(z) = \lambda z^n$, $\lambda$ being a complex number.

Since the inequality (1.1) becomes equality only when $p(z) = \lambda z^n$, that is, when all the zeros of $p(z)$ lie at the origin, it should be possible to improve upon (1.1) when $p(0) \neq 0$, and this has been done by Frappier, Rahman and Ruscheweyh [2] who proved that for polynomials $p(z) = \sum_{v=0}^{n} a_v z^v$ of degree $n \geq 2$,

$$M(p,R) \leq R^n \|p\| - (R^n - R^{n-2})|a_0|, \tag{1.2}$$

for $R \geq 1$.

For polynomials $p(z)$ having no zeros in $|z| < 1$, Ankeny and Rivlin [1] sharpened (1.1) by proving the following

THEOREM A. *If $p(z)$ is a polynomial of degree $n$ and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \frac{R^n + 1}{2} \|p\|. \tag{1.3}$$

The result is best possible and the equality holds for $p(z) = (\lambda + \mu z^n)$, $\lambda$ and $\mu$ being complex numbers with $|\lambda| = |\mu|$.

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A refinement of the above result was given by Govil [3] who proved

THEOREM B. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* $n$, *and* $p(z) \neq 0$ *in* $|z| < 1$, *then for* $R \geq 1$,

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\| - \frac{n}{2}\left(\frac{\|p\|^2 - 4|a_n|^2}{\|p\|}\right)\left\{\frac{(R-1)\|p\|}{\|p\|+2|a_n|} - \ln\left(1+\frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right)\right\}. \tag{1.4}$$

*The result is best possible and equality holds for* $p(z) = (\lambda + \mu z^n)$, $\lambda$ *and* $\mu$ *being complex with* $|\lambda| = |\mu|$.

The bound given in Theorem B is always sharper than the bound given in Theorem A, except when $|a_n| = \|p\|/2$, in which case Theorem B trivially reduces to Theorem A.

Although the inequality (1.4) is best possible, the drawback of this result is that it only depends on the fact that all the zeros are $\geq 1$ in moduli, and not on the particular modulus of each zero. For example, for both polynomials $p(z) = (z+1)^n$ and $q(z) = (z+100)^n$, Govil's result (1.4) will give the same bound even though all the zeros of the polynomial $q(z)$ lie much farther away from the origin than all the zeros of $p(z)$; note that the polynomial $q(z)$ has all its zeros on $|z| = 100$ while $p(z)$ has all on $|z| = 1$. It will therefore be nice to obtain a bound that depends on the moduli of all the zeros of the polynomial and includes as a special case Govil's result [3], and to this effect we prove

THEOREM 1. *If* $p(z) = a_n \Pi_{v=1}^{n}(z - z_v)$, *with* $|z_v| \geq K_v \geq 1$ *for* $v = 1, 2, \ldots, n$ *then for* $R \geq 1$,

$$M(p,R) \leq L\|p\|\left(\frac{R^n-1}{2} + \frac{1}{L}\right) - n\frac{(L^2\|p\|^2 - 4|a_n|^2)}{2L\|p\|}\left\{\frac{(R-1)L\|p\|}{2|a_n| + L\|p\|} - \ln\left(1 + \frac{(R-1)L\|p\|}{2|a_n| + L\|p\|}\right)\right\} \tag{1.5}$$

*where*

$$L = \begin{cases} \left(1 - \frac{1}{1+\frac{2}{n}\sum_{v=1}^{n}\frac{1}{K_v-1}}\right) & \text{if } K_v > 1, \text{ for each } v,\ 1 \leq v \leq n \\ 1 & \text{if } K_v = 1 \text{ for some } v,\ 1 \leq v \leq n. \end{cases}$$

In particular if $K_v = 1$ for some $v$, $1 \leq v \leq n$, then $L = 1$ and as is easy to see then (1.5) reduces to (1.4).

We now shift our attention to the behavior of $M(p,r)$ for $r \leq 1$. From the maximum Modulus Principle, we immediately have that $\|p\| \geq M(p,r) \geq |a_0|$, for $r \leq 1$. If $p(z)$ is a polynomial of degree at most $n$, then so is $q(z) = z^n p(1/z)$. Hence by the Maximum Modulus Principle $M(p,1/r) \geq M(p,1)$ for $0 < r < 1$. Since $M(p,1/r) = r^{-n}M(p,r)$, we obtain that for $0 \leq r \leq 1$,

$$M(p,r) \geq r^n \|p\|. \tag{1.6}$$

In (1.6) equality holds if and only if $p(z) = \lambda z^n$, $\lambda$ being a complex number. It should therefore be possible to improve upon (1.6) if $p(0) \neq 0$, and this was done by Frappier, Rahman and Ruscheweyh [2, p. 92] who proved that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n \geq 2$, then for $0 \leq r \leq 1$,

$$M(p,r) \geq r^n \|p\| + (1-r^2)|a_0|. \tag{1.7}$$

We will prove here a related result:

THEOREM 2. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$, then for $0 \leq r \leq 1$,*

$$M(p,r) \geq \frac{1}{2} r^{n-1}(\|p\| - |a_n|) + \frac{1}{2}\sqrt{r^{2n-2}(\|p\| - |a_n|)^2 + 4r^{2n}|a_n|\|p\|}. \tag{1.8}$$

We do not claim that (1.8) is in general, an improvement of the bound given in (1.6) and (1.7). However in some cases (1.8) can give considerable improvement over the bound obtained from (1.6) and this we show by means of the following example.

EXAMPLE. Let $p(z) = (20-20i)z + (30-50i)z^2 + (1+5i)z^3 + .0001z^4$. Then as is easy to verify, (1.6) gives $M(p,.5) \geq 5.164$ while by (1.8) we have $M(p,.5) \geq 10.329$, an improvement almost by a factor of 2.

## 2. *Lemmas*

For the proof of our theorems, we will need the following lemmas.

LEMMA 1. *For $R \geq 1$ and $n \geq 0$,*

$$f(x) = \left\{1 - \frac{(x - n|a_n|)(R-1)}{(n|a_n| + Rx)}\right\} x, \tag{2.1}$$

*is positive and is an increasing function of $x$ for $x > 0$.*

As is easy to verify, Lemma 1 follows from the first derivative test and is due to Govil [3, p. 81].

LEMMA 2. *If $p(z) = a_n \Pi_{v=1}^{n}(z - z_v)$ is a polynomial of degree $n$ with $|z_v| \geq K_v \geq 1$ for $v = 1, 2, \ldots, n$, then*

$$\|p'\| \leq \begin{cases} n\left(\dfrac{\sum_{v=1}^{n} \frac{1}{K_v - 1}}{\sum_{v=1}^{n} \frac{K_v+1}{K_v-1}}\right)\|p\|, & \text{if } K_v > 1 \text{ for } 1 \leq v \leq n \\ \frac{n}{2}\|p\|, & \text{if } K_v = 1 \text{ for some } v,\ 1 \leq v \leq n. \end{cases} \tag{2.2}$$

Lemma 2 is due to Govil and Labelle [4].

LEMMA 3. *Under the hypotheses of Lemma 2, we have*

$$\|p'\| \leq \begin{cases} \frac{n}{2}\left\{1 - \frac{1}{1+\frac{2}{n}\sum_{v=1}^{n}\frac{1}{K_v-1}}\right\}\|p\|, & \text{if } K_v > 1 \text{ for } 1 \leq v \leq n \\ \frac{n}{2}\|p\|, & \text{if } K_v = 1 \text{ for some } v,\ 1 \leq v \leq n. \end{cases} \tag{2.3}$$

Lemma 3 follows immediately from Lemma 2, by observing that if $K_v > 1$ for $1 \leq v \leq n$, then

$$\frac{\sum_{v=1}^{n}\frac{1}{K_v-1}}{\sum_{v=1}^{n}\frac{K_v+1}{K_v-1}} = \frac{1}{2}\left(1 - \frac{1}{1+\frac{2}{n}\sum_{v=1}^{n}\frac{1}{K_v-1}}\right).$$

LEMMA 4. *Let* $p(z) = \Pi_{v=1}^{n} a_n(z - z_v) = \sum_{v=1}^{n} a_v z^v$ *be a polynomial of degree* $n$ *with* $|z_v| \geq K_v \geq 1$, $1 \leq v \leq n$. *Then*

$$|a_n| \leq \frac{L}{2}\|p\|, \tag{2.4}$$

*where* $L$ *is as defined in the statement of Theorem* 1.

PROOF. Note that $p'(z) = \sum_{v=1}^{n} v a_v z^{v-1}$ and hence by a result due to Visser [7, Lemma 3],

$$|a_1| + |na_n| \leq \|p'\|, \tag{2.5}$$

which implies in particular that

$$|na_n| \leq \|p'\|, \tag{2.6}$$

and this when combined with (2.3) gives (2.4), and the proof of Lemma 4 is thus complete. ∎

LEMMA 5. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* $n$, *then for* $|z| = R \geq 1$,

$$|p(z)| \leq R^n \left\{1 - \frac{(\|p\| - |a_n|)(R-1)}{(|a_n| + R\|p\|)}\right\}\|p\|. \tag{2.7}$$

*This result is best possible and equality holds for* $p(z) = \lambda z^n$, $\lambda$ *being a complex number.*

Lemma 5 is also due to Govil [3, Lemma 3].

## 3. *Proofs of the Theorems*

PROOF OF THEOREM 1. We can assume without loss of generality that $K_v > 1$ for all $v$, $1 \leq v \leq n$, because if $K_v = 1$ for some $v$, $1 \leq v \leq n$, Theorem 1 reduces to Theorem B.

Note that

$$\int_1^R p'(re^{i\phi})e^{i\phi}\,dr = p(Re^{i\phi}) - p(e^{i\phi}),$$

therefore

$$|p(Re^{i\phi}) - p(e^{i\phi})| = \left|\int_1^R p'(re^{i\phi})e^{i\phi}\,dr\right| \leq \int_1^R |p'(re^{i\phi})|\,dr,$$

which gives by Lemma 5, that

$$|p(Re^{i\phi}) - p(e^{i\phi})| \le \int_1^R r^{n-1} \left\{1 - \frac{(\|p'\| - n|a_n|)(r-1)}{(n|a_n| + r\|p'\|)}\right\} \|p'\|\, dr. \tag{3.1}$$

By Lemma 1, the integrand in (3.1) is positive and an increasing function of $\|p'\|$ and so by Lemma 3,

$$\begin{aligned} |p(Re^{i\phi}) - p(e^{i\phi})| &\le \int_1^R r^{n-1} \left\{1 - \frac{(\frac{nL}{2}\|p\| - n|a_n|)(r-1)}{(n|a_n| + \frac{rnL}{2}\|p\|)}\right\} \frac{nL}{2}\|p\|\, dr \\ &= L\|p\|\frac{(R^n-1)}{2} - \frac{nL}{2}\|p\| \left(\frac{L\|p\|}{2} - |a_n|\right) \int_1^R \frac{r^{n-1}(r-1)}{|a_n| + \frac{rL}{2}\|p\|}\, dr. \end{aligned}$$

By Lemma 4, $|a_n| \le \frac{L\|p\|}{2}$, so $\frac{L\|p\|}{2} - |a_n| \ge 0$, and therefore

$$\begin{aligned} |p(Re^{i\phi}) - p(e^{i\phi})| &\le L\|p\|\frac{(R^n-1)}{2} - \frac{nL\|p\|}{2}(L\|p\| - 2|a_n|) \int_1^R \frac{r-1}{2|a_n| + rL\|p\|}\, dr \\ &= L\|p\|\frac{(R^n-1)}{2} - \frac{n}{2}(L\|p\| - 2|a_n|) \int_1^R \left\{1 - \left(\frac{L\|p\| + 2|a_n|}{rL\|p\| + 2|a_n|}\right)\right\} dr \\ &= L\|p\|\frac{(R^n-1)}{2} - \frac{n}{2}(L\|p\| - 2|a_n|) \{(R-1) \\ &\quad - \frac{(L\|p\| + 2|a_n|)}{L\|p\|} \ln\left(\frac{RL\|p\| + 2|a_n|}{L\|p\| + 2|a_n|}\right)\Big\} \\ &= L\|p\|\frac{(R^n-1)}{2} - \frac{n}{2}\frac{(L^2\|p\|^2 - 4|a_n|^2)}{L\|p\|} \left\{\frac{(R-1)L\|p\|}{L\|p\| + 2|a_n|}\right. \\ &\quad \left. - \ln\left(1 + \frac{(R-1)L\|p\|}{2|a_n| + L\|p\|}\right)\right\}. \end{aligned} \tag{3.2}$$

Since $|p(Re^{i\phi})| - \|p\| \le |p(Re^{i\phi}) - p(e^{i\phi})|$, hence we get from (3.2), that

$$\begin{aligned} M(p,R) &\le L\|p\|\left(\frac{R^n-1}{2} + \frac{1}{L}\right) - \frac{n}{2}\frac{(L\|p\|^2 - 4|a_n|^2)}{L\|p\|} \left\{\frac{(R-1)L\|p\|}{L\|p\| + 2|a_n|}\right. \\ &\quad \left. - \ln\left(1 + \frac{(R-1)L\|p\|}{2|a_n| + L\|p\|}\right)\right\}. \end{aligned}$$

which is (1.5) and the proof of Theorem 1 is therefore complete. ∎

**Proof of Theorem 2:** If $0 < r \le 1$ then $R = 1/r \ge 1$. Let $q(z) = p\left(\frac{z}{R}\right)$. Then $M(q,R) = \|p\|$ and $\|q\| = M(p,r)$. Hence by Lemma 5,

$$\begin{aligned} \|p\| = M(q,R) &\le R^n\|q\| - \frac{(\|q\| - \frac{|a_n|}{R^n})(R-1)}{(\frac{|a_n|}{R^n} + R\|q\|)} R^n\|q\| \\ &= R^n \left\{\frac{R|a_n| + R^n M(p,r)}{R^{n+1}M(p,r) + |a_n|}\right\} M(p,r), \end{aligned}$$

which gives

$$\|p\|(R^{n+1}M(p,r)+|a_n|) \le \left(R^{n+1}|a_n| + R^{2n}M(p,r)\right)M(p,r),$$

which is equivalent to

$$R^{2n}\{M(p,r)\}^2 - R^{n+1}(\|p\| - |a_n|)M(p,r) - |a_n|\|p\| \ge 0.$$

The above inequality clearly gives

$$\begin{aligned} M(p,r) &\ge \frac{R^{n+1}(\|p\| - |a_n|) + \sqrt{R^{2n+2}(|a_n| - \|p\|)^2 + 4R^{2n}|a_n|\|p\|}}{2R^{2n}} \\ &= \frac{1}{2}r^{n-1}(\|p\| - |a_n|) + \frac{1}{2}\sqrt{r^{2n-2}(|a_n| - \|p\|)^2 + 4r^{2n}|a_n|\|p\|}, \end{aligned}$$

which completes the proof of Theorem 2. ∎

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