Analysis 17, 395 – 402 (1997)

Analysis

FUNCTIONS OF EXPONENTIAL TYPE NOT VANISHING IN A HALF-PLANE

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Received: August 14, 1996; revised: August 8, 1997

Abstract: If $f(z)$ is an entire function of exponential type, $h_f(\pi/2) = 0$ and $f(z) \neq 0$ for $Im(z) > 0$ then according to a well-known result of R. P. Boas, for $-\infty < x < \infty$, we have $|f'(x)| \leq \frac{\tau}{2} \sup_{-\infty<x<\infty} |f(x)|$. R. P. Boas proposed the problem of obtaining an inequality analogous to this if $f(z) \neq 0$ for $Im(z) > k$, k being real and the answer to this question in the case $k < 0$ was given by Govil and Rahman. In this paper we present generalizations of these results of Govil and Rahman.

AMS (MOS) Mathematics Subject Classification (1991). Primary 30D15; Secondary 30A10, 30E10.

Key words and phrases. Special classes of entire functions and growth estimates, Inequalities in the complex domain, Approximation in the complex domain.

1. Introduction and Statement of Results.

An entire function is said to be of *exponential type* τ if it is of order less than 1 or it is of order 1 and type less than or equal to τ. We will denote this class of functions by $\mathcal{E}_\tau$. For $f \in \mathcal{E}_\tau$, define $\|f\| = \sup_{-\infty<x<\infty} |f(x)|$. The *indicator function* $h_f(\theta)$ of f is defined by

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}.$$

A classical result of Bernstein (see Boas [2, p. 206]) states that if $f \in \mathcal{E}_\tau$ and if $\|f\| = 1$, then

$$(1.1) \qquad \|f'\| \leq \tau.$$

Boas [3] proved that if $h_f\left(\frac{\pi}{2}\right) = 0$ and $f(x+iy) \neq 0$ for $y > 0$, then (1.1) can be replaced by

$$(1.2) \qquad |f'(x)| \leq \frac{\tau}{2}.$$

In connection with the inequality (1.2), Professor R. P. Boas proposed the following problem: "If $f(z)$ is an entire function of exponential type τ such that $|f(x)| \leq 1$ for all real x, $h_f(\pi/2) = 0$ and $f(x+iy) \neq 0$ for $y > k$, then what is the bound for $|f'(x)|$?" For $k = 0$ the answer should reduce to (1.2).

In the problem proposed by Professor Boas, the hypothesis "$f(x+iy) \neq 0$ for $y > k$" is stronger than the hypothesis "$f(x+iy) \neq 0$ for $y > 0$" if $k < 0$, but surprisingly as has been shown by Govil and Rahman [7] the improvement over (1.2) is not possible. However with some additional hypotheses it is possible to improve (1.2) and this has been done in the following results due to Govil and Rahman [7].

THEOREM A. *Let $f(z)$ be an entire function of exponential type τ having all its zeros on $Im(z) = k \leq 0$. If $h_f(\pi/2) = 0$, $h_{f'}(\pi/2) = -c < 0$, and $|f(x)| \leq 1$ for real x, then*

$$(1.3) \qquad |f'(x)| \leq \frac{\tau}{1+\exp(c|k|)}, \quad -\infty < x < \infty.$$

THEOREM B. *Let $f(z)$ be an entire function of exponential type τ having all its zeros in $Im(z) \leq k \leq 0$. If $h_f(\pi/2) = 0$, $h_{f'}(\pi/2) = -c < 0$, and also $h_{\omega'}(\pi/2) \leq -c < 0$ where $\omega(z) = e^{i\tau z}\{\overline{f(\overline{z})}\}$, then $|f(x)| \leq 1$ for real x implies* (1.3).

Both the above results are sharp and the function $f_c(z) = \left(\dfrac{\exp(icz) - \exp(-ck)}{1+\exp(-ck)}\right)^{\tau/c}$ is extremal for both Theorems A and B.

DEFINITION. *For $f \in \mathcal{E}_\tau$, we define with respect to a complex number ζ, the function $D_\zeta[f]$ as $D_\zeta[f] = \tau f(z) + i(1-\zeta)f'(z)$.*

This definition is due to Rahman and Schmeisser [11]. Note that $\lim_{\zeta\to\infty} \dfrac{D_\zeta[f(z)]}{\zeta} = -if'(z)$.

In this paper we generalize the above two theorems of Govil and Rahman [7] by proving:

THEOREM 1. *Let $f(z)$ be an entire function of order 1, type less than or equal to τ,*

$h_f\left(\frac{\pi}{2}\right) = 0$, $h_{D_\zeta[f]}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = 1$, *and suppose* $f(z)$ *has all its zeros on* $Im(z) = k \leq 0$. *Also suppose that the order and type of* f *and* $D_\zeta[f]$ *are the same. Then*

$$\|D_\zeta[f]\| \leq \frac{\tau(|\zeta|+1)}{e^{c|k|}+1} \tag{1.4}$$

where $|\zeta| \geq 1$.

Since f and $D_\zeta[f]$ can neither be of different orders nor of different types (if of the same order) for large ζ, hence if we divide both sides of (1.4) by $|\zeta|$ and make $\zeta \to \infty$ we get Theorem A in the case f is of order 1 and type less than or equal to τ. The case of Theorem 1 when f is order less than 1 follows trivially because in that case f must be a constant. Note that an entire function of order less than 1 cannot be bounded on any line unless it is a constant.

As a generalization of Theorem B, we can prove

THEOREM 2. *Let* f *and* $D_\zeta[f]$ *be entire functions of order* 1 *and type* τ *such that* $h_f\left(\frac{\pi}{2}\right) = 0$, $h_{D_\zeta[f]}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = 1$, $h_{D_\zeta[\omega]}\left(\frac{\pi}{2}\right) \leq -c < 0$ *where* $\omega(z) = e^{i\tau z}\overline{f(\overline{z})}$. *If* $f(z)$ *has all its zeros in* $Im(z) \leq k \leq 0$ *then*

$$\|D_\zeta[f]\| \leq \frac{\tau(|\zeta|+1)}{e^{c|k|}+1} \tag{1.5}$$

where $|\zeta| \geq 1$.

To obtain Theorem B from Theorem 2, just divide both sides of (1.5) by $|\zeta|$ and make $\zeta \to \infty$.

2. Lemmas.

LEMMA 2.1. *Let* $f \in \mathcal{E}_\tau$ *where* $\tau > 0$ *and* $h_f\left(\frac{\pi}{2}\right) = 0$. *Let* H *denote the (open or closed) upper half plane. If* $f(z) \neq 0$ *for* $z \in H$, *then* $D_\zeta[f(z)] \neq 0$ *for* $z \in H$ *and* $|\zeta| \leq 1$.

Lemma 2.1 is due to Rahman and Schmeisser [11].

If we apply Lemma 2.1 to the function $e^{i\tau z}\overline{f(\overline{z})}$, then we easily get

LEMMA 2.2. *Let* $f \in \mathcal{E}_\tau$ *where* $\tau > 0$ *and* $h_f\left(\frac{-\pi}{2}\right) = \tau$. *Let* L *denote the (open or closed) lower half plane. If* $f(z) \neq 0$ *for* $z \in L$, *then* $D_\zeta[f(z)] \neq 0$ *for* $z \in L$ *and* $|\zeta| \geq 1$.

As a consequence of the Phragmen-Lindelöf Theorem (see, for example, p. 3 of [2]), we have ((6.2.4) of [2], also [4], [9], and [10]):

LEMMA 2.3. *If* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) \le c$ *and* $\|f\| \le M$, *then for every* z *with* $y = Im(z) \ge 0$, $|f(z)| \le Me^{cy}$.

LEMMA 2.4. *If* f *is an entire function of order* 1, *type* τ, $\|f\| = M$ *and* $h_f\left(\frac{\pi}{2}\right) \le 0$, *then* $h_f\left(\frac{-\pi}{2}\right) = \tau$.

PROOF. Let $g(z) = e^{-i\tau z/2} f(z)$. Then $h_g\left(\frac{\pi}{2}\right) \le \frac{\tau}{2}$, and $h_g\left(\frac{-\pi}{2}\right) \le \frac{\tau}{2}$, which on applying a result of Boas (see p. 82, line 14 of [2]), gives

$$|h_g(\theta)| \le \frac{\tau}{2}|\sin\theta| \text{ for all } \theta. \tag{2.1}$$

Since $h_g(\theta) \ge \frac{-\tau}{2} + h_f(\theta)$, we get by (2.1) $\frac{\tau}{2} \ge \frac{\tau}{2}|\sin\theta| \ge \frac{-\tau}{2} + h_f(\theta)$, which implies $h_f(\theta) < \tau$ for $\theta \ne \pm\frac{\pi}{2}$. But by hypothesis $h_f\left(\frac{\pi}{2}\right) \le 0$ and f is of type τ and hence $h_f\left(\frac{-\pi}{2}\right) = \tau$. ∎

The following four lemmas are due to Gardner and Govil [5].

LEMMA 2.5. *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{-\pi}{2}\right) = \tau$, $h_f\left(\frac{\pi}{2}\right) \le 0$, *and* $f(z) \ne 0$ *for* $Im(z) \le 0$. *Then* $|f(z)| \ge |\omega(z)|$ *for* $Im(z) \le 0$ *where* $\omega(z) = e^{i\tau z}\overline{f(\overline{z})}$.

LEMMA 2.6. *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{-\pi}{2}\right) = \tau$, $h_f\left(\frac{\pi}{2}\right) \le 0$, $\|f\| = M$ *and* $|f(z)| \ge |\omega(z)|$ *for* $Im(z) \le 0$ *where* $\omega(z) = e^{i\tau z}\overline{f(\overline{z})}$. *Then for* $|\alpha| > 1$, $h_{\omega(z)-\alpha f(z)}\left(\frac{-\pi}{2}\right) = \tau$.

LEMMA 2.7. *Let* $f \in \mathcal{E}_\tau$ *with* $\|f\| = 1$ *and* $h_f\left(\frac{\pi}{2}\right) = 0$. *Then for any* z *with* $Im(z) = y \le 0$ *and* $|\zeta| \ge 1$, *we have* $|D_\zeta[f(z)]| + |D_\zeta[\omega(z)]| \le \tau\left(|\zeta|e^{\tau|y|} + 1\right)$, *where* $\omega(z) = e^{i\tau z}\overline{f(\overline{z})}$.

LEMMA 2.8. *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $\|f\| = 1$ *and* $f(z) \ne 0$ *for* $Im(z) > 0$. *Then* $|D_\zeta[f(z)]| \le \frac{\tau}{2}\left(|\zeta|e^{\tau|y|} + 1\right)$ *for* $y = Im(z) \le 0$ *and* $|\zeta| \ge 1$.

LEMMA 2.9. *Let* $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $\|f\| = M$ *and* $f(z) \ne 0$ *for* $Im(z) > k \ge 0$. *Then for* $-\infty < x < \infty$, *and* $|\zeta| \ge 1$, *we have* $|D_\zeta[f(x)]| \le e^{-\tau k}|D_\zeta[\omega(x-2ik)]|$ *where* $\omega(z) = e^{i\tau z}\overline{f(\overline{z})}$.

PROOF. Let $F(z) = f(z+ik)$. Then $F(z) \neq 0$ for $Im(z) > 0$. Further, if $\Omega(z) = e^{i\tau z}\overline{F(\overline{z})} = e^{-\tau k}\omega(z-ik)$, then $\Omega(z) \neq 0$ for $Im(z) < 0$ and $h_\Omega\left(\frac{-\pi}{2}\right) = \tau + h_f\left(\frac{\pi}{2}\right) = \tau$ and $h_\Omega\left(\frac{\pi}{2}\right) = -\tau + h_f\left(\frac{-\pi}{2}\right) \leq 0$. So by applying Lemma 2.5 to $\Omega(z)$, we get $|\Omega(z)| \geq |F(z)|$ for $Im(z) \leq 0$. So for any α such that $|\alpha| > 1$, $F(z) - \alpha\Omega(z) \neq 0$ in $Im(z) \leq 0$. Also, applying Lemma 2.3, we see that $\|\Omega\| \leq M$ (note that $\|\Omega\| = \|F\| = \sup_{-\infty<x<\infty} |f(x+ik)| \leq M$ because by Lemma 2.3 $|f(z)| \leq M$ for $Im(z) \geq 0$) and so we can apply Lemma 2.6 to $\Omega(z)$ to get $h_{F(z)-\alpha\Omega(z)}\left(\frac{-\pi}{2}\right) = \tau$. Applying Lemma 2.2 to $F(z) - \alpha\Omega(z)$, we get $D_\zeta[F(z) - \alpha\Omega(z)] \neq 0$ for $Im(z) \leq 0$ and $|\zeta| \geq 1$, which implies that $|D_\zeta[F(z)]| \leq |D_\zeta[\Omega(z)]|$ for $Im(z) \leq 0$ and $|\zeta| \geq 1$. In particular we have for $-\infty < x < \infty$ $|D_\zeta[F(x-ik)]| \leq |D_\zeta[\Omega(x-ik)]|$, which implies $|D_\zeta[f(x)]| = |D_\zeta[F(x-ik)]| \leq |D_\zeta[\Omega(x-ik)]| = e^{-\tau k}|D_\zeta[\omega(x-2ik)]|$. ∎

LEMMA 2.10. *Let $f \in \mathcal{E}_\tau$, $h_f\left(\frac{-\pi}{2}\right) = \tau$, $\|f\| = M$, and suppose $f(z)$ has all its zeros in $Im(z) \geq k \geq 0$. Then for $-\infty < x < \infty$ and $|\zeta| \geq 1$, we have $e^{-\tau k}|D_\zeta[\omega(x-2ik)]| \leq |D_\zeta[f(x)]|$ where $\omega(z) = e^{i\tau z}\overline{f(\overline{z})}$.*

PROOF. Since $f(z)$ has all its zeros in $Im(z) \geq k \geq 0$, $\omega(z)$ has all its zeros in $Im(z) \leq -k \leq 0$. So $\omega(z-2ik) \neq 0$ for $Im(z) > k \geq 0$. Also, $h_\omega\left(\frac{\pi}{2}\right) = -\tau + h_f\left(\frac{-\pi}{2}\right) = 0$. Since $e^{i\tau z}\overline{\omega(\overline{z}-2ik)} = e^{2\tau k}f(z+2ik)$, we have, by applying Lemma 2.9 to $\omega(z-2ik)$, that $|D_\zeta[\omega(x-2ik)]| \leq e^{-\tau k}|D_\zeta[e^{2\tau k}\overline{f(x)}]|$, which gives $e^{-\tau k}|D_\zeta[\omega(x-2ik)]| \leq |D_\zeta[f(x)]|$. ∎

LEMMA 2.11. *Let $f \in \mathcal{E}_\tau$, $h_f\left(\frac{\pi}{2}\right) = 0$, $\|f\| = M$, and suppose $f(z) \neq 0$ for $Im(z) > k$ where $k \leq 0$. Then for $-\infty < x < \infty$ and $|\zeta| \geq 1$ we have $e^{\tau k}|D_\zeta[f(x+2ik)]| \leq |D_\zeta[\omega(x)]|$, where $\omega(z) = e^{i\tau z}\overline{f(\overline{z})}$.*

PROOF. Note that $\omega(z)$ has all its zeros in $Im(z) \geq -k \geq 0$. Also, $h_\omega\left(\frac{-\pi}{2}\right) = \tau$. So, applying Lemma 2.10 to $\omega(z)$, we get $e^{-\tau(-k)}|D_\zeta[f(x-2i(-k))]| \leq |D_\zeta[\omega(x)]|$, which gives $e^{\tau k}|D_\zeta[f(x+2ik)]| \leq |D_\zeta[\omega(x)]|$. ∎

The following result is due to Govil and Rahman [7]:

LEMMA 2.12. *Let $f(z)$ be an entire function of order 1 and type τ such that $h_f\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = M$, and suppose $f(z)$ has all its zeros in $Im(z) \geq k$ where $k \leq 0$. Then for $-\infty < x < \infty$ we have $|f(x+2ik)| \geq e^{(\tau+c)|k|}|f(x)|$.*

LEMMA 2.13. *Let f and $D_\zeta[f]$ be entire functions of order 1 and type τ such that $h_f\left(\frac{\pi}{2}\right) = 0$, $h_{D_\zeta[f]}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = M$, and suppose $f(z)$ has all its zeros on $Im(z) = k \leq 0$. Then for $-\infty < x < \infty$ and $|\zeta| \geq 1$, we have $e^{c|k|}|D_\zeta[f(x)]| \leq |D_\zeta[\omega(x)]|$, where $\omega(z) = e^{i\tau z}\overline{f(\overline{z})}$.*

PROOF. Applying Lemma 2.11 to $f(z)$ we get

$$|D_\zeta[\omega(x)]| \geq e^{\tau k}|D_\zeta[f(x+2ik)]|. \tag{2.2}$$

By Lemma 2.4, $h_f\left(\frac{-\pi}{2}\right) = \tau$. So $f(x+2ik)$ satisfies the hypotheses of Lemma 2.2 and hence by applying Lemma 2.2, $D_\zeta[f(x+2ik)]|$ has no zeros in $Im(z) < 0$. So $D_\zeta[f(z)]$ has all its zeros in $Im(z) \geq k$. Also, by Lemma 2.8, $D_\zeta[f(x)]$, where $-\infty < x < \infty$, is bounded by $\frac{\tau}{2}(|\zeta|+1)$ and so by Lemma 2.12 applied to $D_\zeta[f(x)]$,

$$|D_\zeta[f(x+2ik)]| \geq e^{(\tau+c)|k|}|D_\zeta[f(x)]| \tag{2.3}$$

for $-\infty < x < \infty$. Combining (2.2) and (2.3) we get $|D_\zeta[\omega(x)]| \geq e^{c|k|}|D_\zeta[f(x)]|$ for $-\infty < x < \infty$. ∎

LEMMA 2.14. *Let f and $D_\zeta[f]$ be entire functions of order 1 and type τ such that $h_f\left(\frac{\pi}{2}\right) = 0$, $h_{D_\zeta[f]}\left(\frac{\pi}{2}\right) \leq -c < 0$, $\|f\| = M$, $h_{D_\zeta[\omega]}\left(\frac{\pi}{2}\right) \leq -c < 0$ where $\omega(z) = e^{i\tau z}\overline{f(\overline{z})}$. Also, suppose $f(z)$ has all its zeros in $Im(z) \leq k \leq 0$. Then for $-\infty < x < \infty$ and $|\zeta| \geq 1$ we have $e^{c|k|}|D_\zeta[f(x)]| \leq |D_\zeta[\omega(x)]|$.*

PROOF. Let $L_\gamma(z) = f(z) + e^{i\gamma}e^{i\tau(z-ik)}\overline{f(\overline{z}+2ik)}$. As shown by Govil and Rahman [7], $L_\gamma(z)$ satisfies the hypotheses of Lemma 2.13 except possibly for two values of γ (mod 2π). So for $-\infty < x < \infty$ we have

$$e^{c|k|}|D_\zeta[f(x) + e^{i\gamma}e^{i\tau(x-ik)}\overline{f(x+2ik)}]| \leq |D_\zeta[\omega(x) + e^{-i\tau}e^{\tau k}f(x+2ik)]| \tag{2.4}$$

for every $\gamma \in [0, 2\pi)$ except possibly two. By continuity (2.4) holds for all γ. So for $-\infty < x < \infty$ we have

$$e^{c|k|}|D_\zeta[f(x)] + e^{i\gamma}D_\zeta[e^{i\tau(x-ik)}\overline{f(x+2ik)}]| \leq |D_\zeta[\omega(x)] + e^{-i\gamma}e^{\tau k}D_\zeta[f(x+2ik)]| \tag{2.5}$$

for all γ. Now choose γ such that the left-hand side of equation (2.5) is

$$e^{c|k|}|D_\zeta[f(x)]| + e^{c|k|}|D_\zeta[e^{i\tau(x-ik)}\overline{f(x+2ik)}]|,$$

which implies by the Triangle Inequality that

$$(2.6) \qquad e^{c|k|}|D_\zeta[f(x)]| + e^{c|k|}|D_\zeta[e^{i\tau(x-ik)}\overline{f(\bar{x}+2ik)}]| \le |D_\zeta[\omega(x)]| + e^{\tau k}|D_\zeta[f(x+2ik)]|.$$

Now choose γ such that the right-hand side of equation (2.5) is

$$|D_\zeta[\omega(x)]| - e^{\tau k}|D_\zeta[f(x+2ik)]|,$$

which is possible by Lemma 2.11, and implies

$$(2.7) \qquad e^{c|k|}|D_\zeta[f(x)]| - e^{c|k|}|D_\zeta[e^{i\tau(x-ik)}\overline{f(\bar{x}+2ik)}]| \le |D_\zeta[\omega(x)]| - e^{\tau k}|D_\zeta[f(x+2ik)]|.$$

Adding the corresponding sides of equations (2.6) and (2.7) we get

$$2e^{c|k|}|D_\zeta[f(x)]| \le 2|D_\zeta[\omega(x)]|,$$

which is equivalent to the conclusion of the lemma. ∎

3. Proofs of the Theorems.

PROOF OF THEOREM 1. In case f and $D_\zeta[f]$ are both of order 1 and type τ, Theorem 1 follows immediately by combining Lemma 2.7 (with $Im(z) = y = 0$) with Lemma 2.13. In case f is of type less than τ the result holds trivially. ∎

PROOF OF THEOREM 2. Again when f is of order 1 and type τ to obtain Theorem 2, simply combine Lemma 2.7 (with $Im(z) = y = 0$) with Lemma 2.14. When f is of type less than τ, the result again holds trivially. ∎

Acknowledgements

The authors gratefully acknowledge the referee for several useful suggestions.

References

1. Ahlfors, L.: Complex Analysis. New York: McGraw-Hill 1979.

2. Boas, R. P.: Entire Functions. New York: Academic Press 1954.

3. Boas, R. P.: Inequalities for asymmetric entire functions. Illinois J. Math. 1, 94-97 (1957).

4. Duffin, R., Schaeffer, A.: Some properties of functions of exponential type. Bull. Amer. Math. Soc. 44, 236-240 (1938).

5. Gardner, R., Govil, N. K.: Some inequalities for entire functions of exponential type. Proc. Amer. Math. Soc. 123, 2757-2761 (1995).

6. Govil, N. K.: An inequality for functions of exponential type not vanishing in a half-plane. Proc. Amer. Math. Soc. 65, 225-229 (1977).

7. Govil, N. K., Rahman, Q. I.: Functions of exponential type not vanishing in a half-plane and related polynomials. Trans. Amer. Math. Soc. 137, 501-517 (1969).

8. Levin, B.: On a special class of entire functions and on related extremal properties of entire functions of finite degree (Russian). Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 45-84 (1950).

9. Plancherel, M., Polya, G.: Fonctions entieres et integrales de Fourier multiples. Comment. Math. Helv. 9, 224-248 (1937); 10, 110-163 (1938).

10. Polya, G., Szegö, G.: Aufgaben und Lehrsätze aus der Analysis. Berlin: Springer 1925.

11. Rahman, Q. I., Schmeisser, G.: Extension of a theorem of Laguerre to entire functions of exponential type. J. Math. Anal. Applic. 122, 463-468 (1987).

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