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A Bernstein Type L^p Inequality for a Certain Class of Polynomials

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Bernstein's classical theorem states that for a polynomial P of degree at most n, $\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|$. We give related results for polynomials P satisfying the conditions $P'(0) = P''(0) = \cdots = P^{(m-1)}(0) = 0$ and $P(z) \neq 0$ for $|z| < K$, where $K \geq 1$. We give L^p inequalities valid for $0 \leq p \leq \infty$. © 1998 Academic Press

1. INTRODUCTION AND HISTORY

Let $\mathscr{P}_n$ be the linear space of all polynomials over the complex field of degree less than or equal to n. For $P \in \mathscr{P}_n$, define

$$\|P\|_0 = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| d\theta\right),$$

$$\|P\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right)^{1/p} \qquad \text{for } 0 < p < \infty,$$

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and

$$\|P\|_\infty = \max_{|z|=1} |P(z)|.$$

Notice that $\|P\|_0 = \lim_{p\to 0^+}\|P\|_p$ and $\|P\|_\infty = \lim_{p\to\infty}\|P\|_p$. For $1 \le p \le \infty$, $\|\cdot\|_p$ is a norm (and therefore $\mathscr{P}_n$ is a normed linear space under $\|\cdot\|_p$). However, for $0 \le p < 1$, $\|\cdot\|_p$ does not satisfy the triangle inequality and is therefore not a norm (this follows from Minkowski's inequality—see [10] for details).

Bernstein's well known result relating the supremum norm of a polynomial and its derivative states that if $P \in \mathscr{P}_n$ then $\|P'\|_\infty \le n\|P\|_\infty$ [2]. This inequality reduces to equality if and only if $P(z) = \alpha z^n$ for some complex constant α. Erdös conjectured and Lax proved [6]:

THEOREM 1.1. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, then*

$$\|P'\|_\infty \le \frac{n}{2}\|P\|_\infty.$$

Malik generalized Theorem 1.1 and proved [7]:

THEOREM 1.2. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ for $|z| < K$ where $K \ge 1$, then*

$$\|P'\|_\infty \le \frac{n}{1+K}\|P\|_\infty.$$

Of course, Theorem 1.1 follows from Theorem 1.2 when $K = 1$. Chan and Malik [3] introduced the class of polynomials of the form $P(z) = a_0 + \sum_{v=m}^{n} a_v z^v$. We denote the linear space of all such polynomials as $\mathscr{P}_{n,m}$. Notice that $\mathscr{P}_{n,1} = \mathscr{P}_n$. Chan and Malik presented the following result [3]:

THEOREM 1.3. *If $P \in \mathscr{P}_{n,m}$ and $P(z) \neq 0$ for $|z| < K$ where $K \ge 1$, then*

$$\|P'\|_\infty \le \frac{n}{1+K^m}\|P\|_\infty.$$

Qazi, independently of Chan and Malik, presented the following result which includes Theorem 1.3 [8]:

THEOREM 1.4. *If $P(z) = a_0 + \sum_{v=m}^{n} a_v z^v \in \mathscr{P}_{n,m}$ and $P(z) \neq 0$ for $|z| < K$ where $K \ge 1$, then*

$$\|P'\|_\infty \le \frac{n}{1+s_0}\|P\|_\infty,$$

where

$$s_0 = K^{m+1}\left(\frac{m|a_m|K^{m-1} + n|a_0|}{n|a_0| + m|a_m|K^{m+1}}\right).$$

Since $m|a_m|K^m \leq n|a_0|$, Theorem 1.4 implies Theorem 1.3 (see [8] for details).

Zygmund [11] extended Bernstein's result to L^p norms. DeBruijn [4] extended Theorem 1.1 to L^p norms by showing:

THEOREM 1.5. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, then for $1 \leq p \leq \infty$*

$$\|P'\|_p \leq \frac{n}{\|1 + z\|_p}\|P\|_p.$$

Of course, Theorem 1.5 reduces to Theorem 1.1 with $p = \infty$. Rahman and Schmeisser [9] proved that Theorem 1.5 in fact holds for $0 \leq p \leq \infty$. The purpose of this paper is to show that Theorems 1.3 and 1.4 can be extended to L^p inequalities where $0 \leq p \leq \infty$.

2. STATEMENT OF RESULTS

Our main result is:

THEOREM 2.1. *If $P(z) = a_0 + \sum_{v=m}^{n} a_v z^v \in \mathscr{P}_{n,m}$ and $P(z) \neq 0$ for $|z| < K$ where $K \geq 1$, then for $0 \leq p \leq \infty$*

$$\|P'\|_p \leq \frac{n}{\|s_0 + z\|_p}\|P\|_p,$$

where s_0 is as given in Theorem 1.4.

With $p = \infty$, Theorem 2.1 reduces to Theorem 1.4. As mentioned in Section 1, we can deduce:

COROLLARY 2.2. *If $P \in \mathscr{P}_{n,m}$ and $P(z) \neq 0$ for $|z| < K$ where $K \geq 1$, then for $0 \leq p \leq \infty$*

$$\|P'\|_p \leq \frac{n}{\|K^m + z\|_p}\|P\|_p.$$

With $p = \infty$, Corollary 2.2 reduces to Theorem 1.3.

Of special interest, is the fact that Theorem 2.1 and Corollary 2.2 hold for L^p *norms* for all $1 \leq p \leq \infty$. In particular, we have:

COROLLARY 2.3. *If $P \in \mathscr{P}_{n,m}$ and $P(z) \neq 0$ for $|z| < K$ where $K \geq 1$, then for $1 \leq p \leq \infty$*

$$\|P'\|_p \leq \frac{n}{\|K^m + z\|_p} \|P\|_p.$$

With $m = 1$, Corollary 2.3 yields an L^p version of Theorem 1.2. With $p = \infty$, Corollary 2.3 reduces to Theorem 1.3. With $m = 1$ and $p = \infty$, Corollary 2.3 reduces to Theorem 1.2. Finally, with $m = 1$, $p = \infty$, and $K = 1$, Corollary 2.3 reduces to Theorem 1.1.

3. LEMMAS

We need the following lemmas for the proof of our theorem.

LEMMA 3.1. *If the polynomial $P(z)$ of degree n has no roots in the circular domain C and if $\zeta \in C$ then $(\zeta - z)P'(z) + nP(z) \neq 0$ for $z \in C$.*

Lemma 3.1 is due to Laguerre [5].

DEFINITION 3.2. For $\gamma = (\gamma_0, \ldots, \gamma_n) \in \mathbf{C}^{n+1}$ and $P(z) = \sum_{v=0}^{n} c_v z^v$, define

$$\Lambda_\gamma P(z) = \sum_{v=0}^{n} \gamma_v c_v z^v.$$

The operator Λ_γ is said to be *admissible* if it preserves one of the following properties:

(a) $P(z)$ has all its zeros in $\{z \in \mathbf{C} : |z| \leq 1\}$,

(b) $P(z)$ has all its zeros in $\{z \in \mathbf{C} : |z| \geq 1\}$.

The proof of Lemma 3.3 was given by Arestov [1]:

LEMMA 3.3. *Let $\phi(x) = \psi(\log x)$ where ψ is a convex non-decreasing function on $\mathbf{R}$. Then for all $P(z) \in \mathscr{P}_n$ and each admissible operator Λ_γ*

$$\int_0^{2\pi} \phi(|\Lambda_\gamma P(e^{i\theta})|)\, d\theta \leq \int_0^{2\pi} \phi(c(\gamma, n)|P(e^{i\theta})|)\, d\theta,$$

where $c(\gamma, n) = \max(|\gamma_0|, |\gamma_n|)$.

Qazi proved [8]:

LEMMA 3.4. *If* $P(z) = c_0 + \sum_{v=m}^{n} c_v z^v$ *has no zeros in* $|z| < K$, $K \geq 1$ *then for* $|z| = 1$

$$K^m |P'(z)| \leq s_0 |P'(z)| \leq |Q'(z)|,$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$ *and* s_0 *is as defined in Theorem* 1.4.

4. PROOF OF THEOREM 2.1

By Lemma 3.1 we have $nP(z) - (z - \zeta)P'(z) \neq 0$ for $|z| \leq 1$, $\zeta \leq 1$. Therefore, setting $\zeta = -ze^{-i\alpha}$, $\alpha \in \mathbf{R}$, the operator Λ defined by

$$\Lambda P(z) = (e^{i\alpha} + 1) zP'(z) - ne^{i\alpha} p(z)$$

is admissible and so by Lemma 3.3 with $\psi(x) = e^{px}$,

$$\int_0^{2\pi} \left| (e^{i\alpha} + 1) \frac{dP(e^{i\theta})}{d\theta} - ine^{i\alpha} P(e^{i\alpha}) \right|^p d\theta \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta$$

for $p > 0$. Then

$$\int_0^{2\pi} \left| \frac{dP(e^{i\theta})}{d\theta} + e^{i\alpha} \left\{ \frac{dP(e^{i\theta})}{d\theta} - inP(e^{i\theta}) \right\} \right|^p d\theta \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta.$$

This gives

$$\int_0^{2\pi} \int_0^{2\pi} \left| \frac{dP(e^{i\theta})}{d\theta} + e^{i\alpha} \left\{ \frac{dP(e^{i\theta})}{d\theta} - inp(e^{i\theta}) \right\} \right|^p d\theta \, d\alpha$$

$$\leq 2\pi n^p \int_0^{2\pi} |p(e^{i\theta})|^p \, d\theta. \tag{4.1}$$

Now

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi}\left|\frac{dP(e^{i\theta})}{d\theta}+e^{i\alpha}\left\{\frac{dP(e^{i\theta})}{d\theta}-inP(e^{i\theta})\right\}\right|^p d\theta\, d\alpha\\
&=\int_0^{2\pi}\left|\frac{dP(e^{i\theta})}{d\theta}\right|^p\int_0^{2\pi}\left|1+e^{i\alpha}\left\{\frac{dP(e^{i\theta})/d\theta-inP(e^{i\theta})}{dP(e^{i\theta})/d\theta}\right\}\right|^p d\alpha\, d\theta\\
&=\int_0^{2\pi}\left|\frac{dP(e^{i\theta})}{d\theta}\right|^p\int_0^{2\pi}\left|e^{i\alpha}+\left|\frac{dP(e^{i\theta})/d\theta-inP(e^{i\theta})}{dP(e^{i\theta})/d\theta}\right|\right|^p d\alpha\, d\theta\\
&=\int_0^{2\pi}\left|\frac{dP(e^{i\theta})}{d\theta}\right|^p\int_0^{2\pi}\left|e^{i\alpha}+\left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})}\right|\right|^p d\alpha\, d\theta\\
&\geq\int_0^{2\pi}\left|\frac{dP(e^{i\theta})}{d\theta}\right|^p\int_0^{2\pi}|e^{i\alpha}+s_0|^p\, d\alpha\, d\theta \qquad \text{by Lemma 3.4} \qquad (4.2)
\end{aligned}$$

by the fact that $|e^{i\alpha}+r|$ is an increasing function of r for $r \geq 1$. Thus combining (4.1) and (4.2) we see that

$$\left(\int_0^{2\pi}\left|\frac{dP(e^{i\alpha})}{d\theta}\right|^p d\theta\right)\left(\int_0^{2\pi}|e^{i\alpha}+s_0|^p\, d\alpha\right) \leq 2\pi n^p\int_0^{2\pi}|P(e^{i\theta})|^p\, d\theta$$

from which the theorem follows for $0 < p < \infty$. The result holds for $p = 0$ and $p = \infty$ by letting $p \to 0^+$ and $p \to \infty$, respectively.

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