Bernstein Inequalities for Polynomials

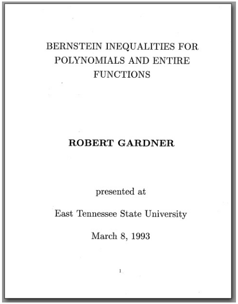
Robert "Dr. Bob" Gardner

March 8, 2013 (Revised 12/24/2017)

(Prepared in Beamer!)

May 8, 1993

In 1993, I was desperate to get out of Louisiana and find a university position in a mountainous region of the country. On May 8, 1993 I interviewed here at ETSU for an assistant professor position. I gave a presentation on "Bernstein Inequalities for Polynomials and Other Entire Functions." This talk on the 20th anniversary of my interview reviews the polynomial results and gives some updates on my work here at ETSU.


BERNSTEIN INEQUALITIES FOR POLYNOMIALS AND ENTIRE FUNCTIONS

ROBERT GARDNER

presented at

East Tennessee State University

March 8, 1993

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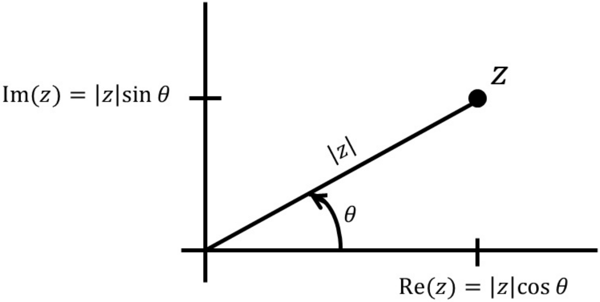
The Complex Plane

The field of complex numbers consists of ordered pairs of real numbers, $\mathbb{C} = \{(a,b) \mid a,b \in \mathbb{R}\}$, with addition defined as $(a,b) + (c,d) = (a+c, b+d)$ and multiplication defined as $(a,b)\cdot(c,d) = (ac - bd, bc + ad)$. We denote (a,b) as $a + ib$. The *modulus* of $z = a + ib$ is $|z| = \sqrt{a^2+b^2}$.



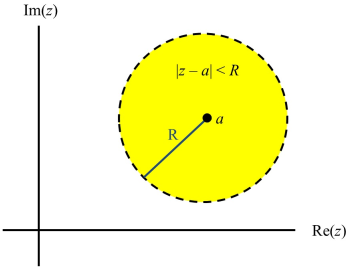
Analytic Functions

Definition

A function $f : G \to \mathbb{C}$, where G is an open connected subset of $\mathbb{C}$, is *analytic* if f is continuously differentiable on G.

Theorem

If f is analytic in $|z-a|<R$ then $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z-a|<R$, where the radius of convergence is at least R.



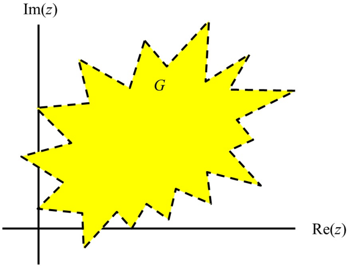
Maximum Modulus Theorem.

Theorem

Let G be a bounded open set in $\mathbb{C}$ and suppose f is continuous on the closure of G, $cl(G)$, and analytic in G. Then

$$\max\{|f(z)| \mid z \in cl(G)\} = \max\{|f(z)| \mid z \in \partial G\}.$$

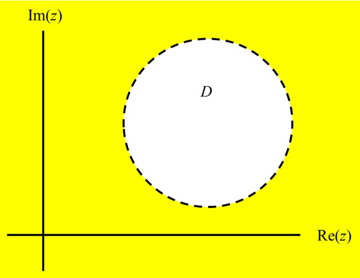
In addition, if $\max\{|f(z)| \mid z \in G\} = \max\{|f(z)| \mid z \in \partial G\}$, *then f is constant.*



The Maximum Modulus Theorem for Unbounded Domains.

Theorem

Let D be an open disk in the complex plane. Suppose f is analytic on the complement of D, continuous on the boundary of D, $|f(z)| \leq M$ on ∂D, and $\lim_{|z|\to\infty} f(z) = M$. Then $|f(z)| \leq M$ on the complement of D.



Lucas' Theorem.

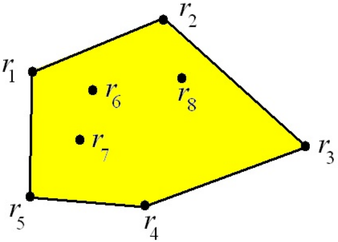
Theorem

If all the zeros of a polynomial P lie in a half plane in the complex plane, then all the zeros of the derivative P' lie in the same half plane.

Corollary

The convex polygon in the complex plane which contains all the zeros of a polynomial P, also contains all the zeros of P'.

Fançois Lucas (1842–1891)



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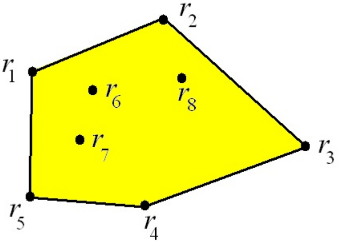
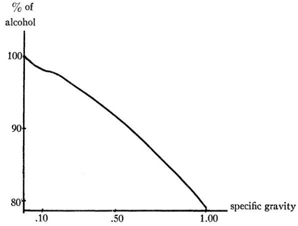


Mendeleev's Data

Russian chemist Dmitri Mendeleev studied the specific gravity of a solution as a function of the percentage of dissolved substance. His data could be closely approximated by quadratic arcs and he wondered if the corners where the arcs joined were real, or due to errors of measurement. His question, after normalization is: "If $p(x)$ is a quadratic polynomial with real coefficients and $|p(x)| \leq 1$ on $[-1, 1]$, then how large can $|p'(x)|$ be on $[-1, 1]$?"



A graph similar to Mendeleev's

Dmitri Mendeleev (1834–1907)

Markov's Result

Mendeleev answered his own question and showed that $|p'(x)| \leq 4$ (and the corners were determined to be genuine). Mendeleev told A. A. Markov about his result, and Markov went on to prove the following.

Theorem

If $p(x)$ is a real polynomial of degree n, and $|p(x)| \leq 1$ on $[-1, 1]$ then $|p'(x)| \leq n^2$ on $[-1, 1]$. Equality holds only at ± 1 and only when $p(x) = \pm T_n(x)$, where $T_n(x)$ is the Chebyshev polynomial $\cos(n \cos^{-1} x)$.

Andrei Andreyevich Markov (1856–1922)

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Such Inequalities in $\mathbb{C}$

We want to address similar inequalities for polynomials over the complex field.

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In this direction, we prove the following lemmas.

Bernstein's Lemma 1.

Theorem

Rate of Growth, Bernstein. *If p is a polynomial of degree n such that $|p(z)| \leq M$ on $|z| = 1$, then for $R \geq 1$ we have*

$$\max_{|z|=R} |p(z)| \leq MR^n.$$

Proof. For $p(z) = \sum_{k=0}^n a_k z^k$ we have $r(z) = z^n p(1/z) = \sum_{k=0}^n a_k z^{n-k}$. Notice that for $|z| = 1$ (and $1/z = \overline{z}$) we have $\|r\| = \|p\|$ where $\|p\| = \max_{|z|=1} |p(z)|$. By the Maximum Modulus Theorem, for $|z| \leq 1$ we have $|r(z)| \leq \|r\| = \|p\| \leq M$. That is, $|z^n p(1/z)| \leq M$ for $|z| \leq 1$. Replacing z with $1/z$, we have $|(1/z^n)p(z)| \leq M$ for $|z| \geq 1$, or $|p(z)| \leq M|z|^n$ for $|z| \geq 1$. □

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Bernstein's Lemma 2.

Lemma

Let P and Q be polynomials such that (i) $\lim_{|z|\to\infty} |p(z)/q(z)| \le 1$*, (ii)* $|P(z)| \le |Q(z)|$ *for* $|z| \le 1$*, and (iii) all zeros of Q lie in* $|z| \le 1$*. Then*

$$|P'(z)| \le |Q'(z)| \text{ for } |z| = 1.$$

Proof. Define $f(z) = P(z)/Q(z)$. Then f is analytic on $|z| > 1$, $|f(z)| \le 1$ for $|z| = 1$, and $\lim_{|z|\to\infty} |f(z)| \le 1$.
So by the Maximum Modulus Theorem for Unbounded Domains,

$$|f(z)| \le 1 \text{ for } |z| \ge 1. \ (*)$$

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Let $|\lambda| > 1$ and define polynomial $g(z) = P(z) - \lambda Q(z)$. If $g(z_0) = P(z_0) - \lambda Q(z_0) = 0$ and if $Q(z_0) \ne 0$ then

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Bernstein and de Bruijn

Bernstein's Lemma was proved (as stated) by Sergei Bernstein [*Leçons sur les propiétés extrémales* (Collection Borel) Paris, 1926]. A generalization was proven by Nicolass de Bruijn [Inequalities concerning polynomials in the complex domain, *Nederl. Akad. Wetensch. Proc.* [*Proceedings of the Royal Dutch Academy of Sciences*] **50** (1947), 1265–1272; *Indagationes Mathematicae, Series A* **9** (1947), 591–598] where $|z| < 1$ is replaced with a convex domain D (a *domain* is an open connected set) and $|z| = 1$ is replaced with the boundary of D, ∂D.

Sergei N. Bernstein (1880–1968)

Nicolaas de Bruijn (1918–2012)

Bernstein's Theorem.

Theorem

Let P be a polynomial of degree n. Then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$

Proof. Let $M = \max_{|z|=1} |P(z)|$ and define $Q(z) = Mz^n$.

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Max Norms

For reasons to be made apparent later, we introduce the following notation:

$$\|P\|_\infty = \max_{|z|=1} |P(z)|.$$

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Let P be a polynomial of degree n with $P(z) \neq 0$ in $|z| < 1$. Then

$$\|P'\|_\infty \leq \frac{n}{2}\|P\|_\infty.$$

Note. This result was conjectured by Erdős and proved by Lax. It is sharp for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

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Govil-Labelle Theorem

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If $P(z) = a_n \prod_{v=1}^{n}(z - z_v)$ $(a_n \neq 0)$*, and* $|z_v| \geq K_v \geq 1$*. Then*

$$\|P'\|_\infty \leq \frac{n}{2}\left(1 - \frac{1}{1 + \frac{2}{n}\sum_{v=1}^{n}\frac{1}{K_v - 1}}\right)\|P\|_\infty.$$

Reference. Narendra K. Govil and Gilbert Labelle, On Bernstein's Inequality, *Journal of Mathematical Analysis and Applications*, **126**(2), 494–500 (1987).

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$$\|P'\|_\infty \leq \frac{n}{2}\left(1 - \frac{1}{1 + \frac{2}{n}\sum_{v=1}^{n}\frac{1}{K_v - 1}}\right)\|P\|_\infty.$$

Reference. Narendra K. Govil and Gilbert Labelle, On Bernstein's Inequality, *Journal of Mathematical Analysis and Applications*, **126**(2), 494–500 (1987).

Note. This result reduces to Malik's Theorem if $K_v \geq K \geq 1$ for all v, and reduces to the Erdős-Lax Theorem if $K_v = 1$ for some v.

Govil-Labelle Theorem

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Professor Qazi Ibadur Rahman

Govil, Malik, and Labelle are each Ph.D. students of Professor Qazi I. Rahman of the University of Montreal: Mohammad Malik (Ph.D. 1967), Narendra Govil (Ph.D. 1968), and Gilbert Labelle (Ph.D. 1969).

Narendra Govil, Qazi Rahman, and Robert Gardner at the 2005 Fall Southeast Section AMS Meeting, October 2005, at ETSU.

Normed Linear Spaces

Definition

Let X be a set of [equivalence classes of] functions. Then X is a *linear space* if for all $f, g \in X$ [or for all equivalence classes $[f], [g] \in X$] and $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in X$ [or $\alpha[f] + \beta[g] \in X$].

Definition

Let X be a linear space. A real-valued functional (i.e., a function with X as its domain and $\mathbb{R}$ as its codomain) $\| \cdot \|$ on X is a *norm* if for all $f, g \in X$ and for all $\alpha \in \mathbb{R}$:

(1) $\|f + g\| \leq \|f\| + \|g\|$ (Triangle Inequality).

(2) $\|\alpha f\| = |\alpha|\|f\|$ (Positive Homogeneity).

(3) $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$.

Real L^p Spaces

Definition

A *normed linear space* is a linear space X with a norm $\|\cdot\|$.

Definition

Let E be a measurable set of real numbers and let $1 \leq p < \infty$. Define $L^p(E)$ to be the set of [equivalence classes of] functions for which $\displaystyle\int_E |f|^p < \infty$.

Definition

For measurable set E, $L^p(E)$ is a normed linear space with *norm* $\|f\|_p = \left\{\displaystyle\int_E |f|^p <\right\}^{1/p}$. In fact, $L^p(E)$ is a complete normed linear space (i.e., a *Banach space*).

L^p Norm

Definition

Let P be a complex polynomial. For $p \geq 1$ the L^p *norm* of P is

$$\|P\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{1/p}.$$

Theorem

If we let $p \to \infty$, then we find that

$$\lim_{p\to\infty} \|P\|_p = \lim_{p\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{1/p} = \max_{|z|=1} |P'(z)| \equiv \|P\|_\infty.$$

Zygmund-Arestov Theorem

Theorem

Let P be a polynomial of degree n. Then for $1 \leq p \leq \infty$, $\|P'\|_p \leq n\|P\|_p$.

References. A. Zygmund, A Remark on Conjugate Series, *Proceedings of the London Mathematical Society* (2) **34**, 392–400 (1932). V. V. Arestov, On Inequalities for Trigonometric Polynomials and Their Derivatives, *Izv. Akad. Nauk SSSR Ser. Mat.* **45**, 3–22 (1981) [in Russian]; *Math. USSR-Izv.* **18**, 1–17 (1982) [in English].

DeBruijn's Theorem.

Theorem

Let P be a polynomial of degree n with $P(z) \neq 0$ in $|z| < 1$. Then for $1 \leq p \leq \infty$,

$$\|P'\|_p \leq \frac{n}{\|1+z\|_p}\|P\|_p.$$

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Note. If we let $p \to \infty$, DeBruijn's Theorem reduces to the Erdös-Lax Theorem.

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If $P(z) = a_n \prod_{v=1}^{n}(z - z_v)$ *(*$a_n \neq 0$*), and* $|z_v| \geq K_v \geq 1$*. Then for* $p \geq 1$*,*

$$\|P'\|_p \leq \frac{n}{\|t_0 + z\|_p}\|P\|_p$$

where $t_0 = \left(\frac{\sum_{v=1}^{n} \frac{K_v}{K_v - 1}}{\sum_{v=1}^{n} \frac{1}{K_v - 1}}\right) = 1 + \frac{n}{\sum_{v=1}^{n} \frac{1}{K_v - 1}}$.

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Note. If $K_v = 1$ for any v, this result reduces to Debruijn's Theorem. If $p \to \infty$, then this result reduces to the Govil-Labelle Theorem.

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L^p "Quantities", $0 \le p < 1$

Definition

Let P be a polynomial. For $0 < p < 1$ define the "L^p *quantity*" of P as

$$\|P\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{1/p}.$$

Note. In the event that we let $p \to 0^+$, we find that

$$\lim_{p \to 0^+} \|P\|_p = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta})| \, d\theta \right).$$

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Note. For $0 \le p < 1$, the L^p quantity of a polynomial does not satisfy the Triangle Inequality. In fact (see Halsey Royden's *Real Analysis*, 3rd Edition, page 120): $\|f + g\|_p \ge \|f\|_p + \|g\|_p$.

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Note. With some $K_v = 1$, we can extend DeBruijn to $0 \leq p < 1$: Let P be a polynomial of degree n with $P(z) \neq 0$ in $|z| < 1$. Then for $0 \leq p \leq \infty$,

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A New Linear Space of Polynomials

Definition

Let $\mathcal{P}_{n,m}$ denote the set of all polynomials of the form

$$P(z) = a_0 + \sum_{v=m}^{n} a_v z^v = a_0 + a_m z^m + a_{m+1} z^{m+1} + \cdots + a_n z^n.$$

Note. $\mathcal{P}_{n,m}$ is a linear space and $\mathcal{P}_{n,1} = \mathcal{P}_n$.

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If $P \in \mathcal{P}_{n,m}$ *and* $P(z) \neq 0$ *for* $|z| < K$ *where* $K \geq 1$, *then*

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If $P(z) = a_0 + \sum_{v=m}^{n} a_v z^v \in \mathcal{P}_{n,m}$ *and* $P(z) \neq 0$ *for* $|z| < K$ *where* $K \geq 1$*, then*

$$\|P'\|_\infty \leq \frac{n}{1+s_0}\|P\|_\infty,$$

where $s_0 = K^{m+1}\left(\frac{m|a_m|K^{m-1} + n|a_0|}{n|a_0| + m|a_m|K^{m+1}}\right)$.

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Note. Qazi worked independently of Chan and Malik. Since $m|a_m|K^m \leq n|a_0|$ (as Qazi explains in his paper), this result implies the Chan-Malik Theorem.

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Gardner-Weems Corollary

Note. With $p = \infty$, this reduces to Qazi's Theorem, and further reduces to the Chan-Malik Theorem. If $p = \infty$ and $m = 1$ this reduces to Malik's Theorem, and if in addition $K = 1$ then it reduces to the Erdős-Lax Theorem.

Corollary

If $P \in \mathcal{P}_{n,m}$ and $P(z) \neq 0$ for $|z| < K$ where $K \geq 1$, then for $1 \leq p \leq \infty$

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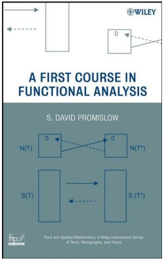
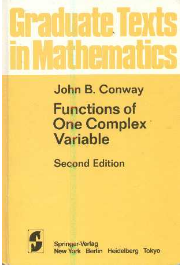
Upcoming Classes

ETSU will offer **Introduction to Functional Analysis** (MATH 5740) during Summer 2013 Term 1. Linear spaces and L^p norms will be a central topic. The prerequisite is Analysis 1 (MATH 4217/5217). During academic 2013–14, **Complex Analysis 1 and 2** (MATH 5510/5520) will be offered. Analytic functions and complex integration will be covered in part 1. The prerequisite is also Analysis 1 and no previous experience with complex variables is assumed. Complex Analysis 2 will cover the Maximum Modulus Theorem and analytic continuation.





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Photographs of mathematicians are primarily from http://www-history.mcs.st-and.ac.uk/.

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