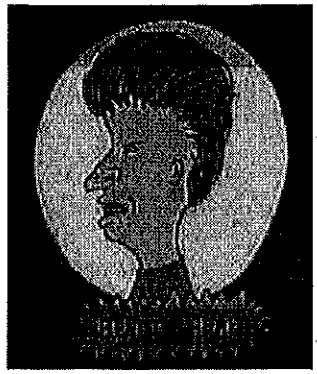
BUTT-HEAD Seminar



I. Vector Spaces: Introduction, Examples, Dimension and Isomorphism

(From "Introduction to Hilbert Spaces with Applications"
by L. Debnath and R. Mikusiński.)

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1.1 VECTOR SPACES - INTRODUCTION

Note. A vector space consists of two things: *scalars* and *vectors*. We take as our scalar field either the real numbers **R** or the complex numbers **C**.

Definition. A *vector space* consists of two nonempty sets: a set V of *vectors* and a set F of *scalars*. An operation called *vector addition* is defined on E and an operation called *scalar multiplication* allows us to multiply a vector by a scalar. The set V is *closed* under these operations. That is,

1. if $x, y \in V$ then $x + y \in V$, and

2. if $\lambda \in F$ and $x \in V$, then $\lambda x \in V$.

We also require the following conditions:

(a) $x + y = y + x$ for all $x, y \in V$

(b) $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$

(c) For all $x, y \in V$, there exists $z \in V$ such that $x + z = y$

(d) $\alpha(\beta x) = (\alpha\beta)x$ for all $\alpha, \beta \in F$ and for all $x \in V$

(e) $(\alpha + \beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in F$ and for all $x \in V$

(f) $\alpha(x + y) = \alpha x + \alpha y$ for all $\alpha \in F$ and for all $x, y \in V$

(g) $1x = x$ for all $x \in V$.

Note Some consequences of the definition of vector spaces are:

1. There exists a unique vector 0 called the *zero vector* such that $0 + x = x$ for all $x \in V$.

2. For each $x \in V$, there exists a unique $y \in V$ such that $x + y = 0$. This y is denoted $-x$ and equals $-1x$.

3. If $\lambda x = 0$ then either $\lambda = 0$ or $x = 0$.

1.2 VECTOR SPACES - EXAMPLES

Example. $n-$dimensional Euclidean space $\mathbf{R}^n$ is a vector space. With $n = 2$ or 3, this yields the familiar idea of vectors as "arrows" which represent position, velocity, or acceleration in introductory physics and engineering classes. In general, elements of $\mathbf{R}^n$ lool like $x = (x_1, x_2, \ldots, x_n)$ where each $x_i \in \mathbf{R}$ (we take the scalar field to be $\mathbf{R}$).

Example. $\mathbf{C}^n = \{(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbf{C}\}$ forms a complex vector space (we take the scalar field to be $\mathbf{C}$). Notice that the "arrows" interpretation is more difficult here (at least for $n > 1$).

Note. You are probably most familiar with the vector spaces $\mathbf{R}^n$ and $\mathbf{C}^n$ and their associated "arrows" interpretations. To make the transition to **Hilbert spaces**, you need to broaden your idea of what a vector is!

Example. The collection of all polynomials of degree n or less forms a vector space of dimension $n+1$ (we can take real or complex polynomials) denoted $\mathcal{P}_n$.

Notice. There is a "natural relationship" between $\mathcal{P}_n$ and $\mathbf{R}^{n+1}$. For Example, we can associate with the polynomial $p(x) = a_0 + a_1x + a_2x^2 \in \mathcal{P}_2$ the element $(a_0, a_1, a_2) \in \mathbf{R}^3$.

Example. The collection of all continuous functions (on $\mathbf{R}$) forms a vector space. Notice that if f and g are continuous, then for all $\alpha, \beta \in \mathbf{R}$, $\alpha f + \beta g$ is continuous. This is, in some sense, a much more complicated vector space than a space of polynomials.

Definition. Let V be a vector space and let

$$x_1, x_2, \ldots, x_k \in V.$$

A *linear combination* of these vectors is a sum of the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are scalars.

Definition. A finite collection of vectors

$$\{x_1, x_2, \ldots, x_k\}$$

is *linearly independent* if

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = 0$$

implies that $\alpha_1 = \alpha_2 = \cdots \alpha_k = 0$. A collection of vectors which is not linearly independent, is *linearly dependent.*

Notice. The vectors $(1,0,0), (0,1,0), (0,0,1)$ are linearly independent vectors in $\mathbf{R}^3$. The vectors $1, x, x^2$ are linearly independent vectors in $\mathcal{P}_2$.

Note. If a mass m is put on a spring (with spring constant k) which is suspended vertically and displaced, then its motion is described by the second order linear differential equation

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

There are two "fundamental" solutions to this DE: $x(t) = \cos \omega t$ and $x(t) = \sin \omega t$, where $\omega = \sqrt{k/m}$. In fact, any linear combination of these two solutions is again a solution and the general solution is $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$. In fact, the collection of all solutions of the DE form a vector space which is "generated" by the sin and cos. (Of course, this is *simple harmonic motion.*)

Definition. The *span* of a finite set of vectors

$$\{x_1, x_2, \ldots, x_k\}$$

is the collection of all possible linear combinations of the vectors:

$$\text{span}(\{x_1, x_2, \ldots, x_k\}) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in F(\text{the scalar field})\}.$$

Note. The span of a set of vectors is a vector space (algebraic properties are inherited from the "larger" vector space, and closure follows from the definition of span).

Note. The collection of all possible solutions of $\dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = 0$ is the vector space $\text{span}(\cos \omega t, \sin \omega t)$.

Definition. A set of vectors $\mathcal{B} \subset V$ is a *basis of* V if $\mathcal{B}$ is linearly independent and span $\mathcal{B} = V$. If a vector space has a finite basis, it is *finite dimensional.* Otherwise, it is *infinite dimensional.*

Note. The vector space of all possible solutions of $\dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = 0$ has as a basis $\{\cos\omega t, \sin\omega t\}$ (notice that this basis *is not unique*!). Therefore, this **second** order linear homogeneous ODE has a **two** dimensional vector space of solutions.

Note. The vector space $\mathbf{R}^3$ has as a basis

$$\{(1,0,0),(0,1,0),(0,0,1)\}.$$

The vector space $\mathcal{P}_2$ has as a basis $\{1, x, x^2\}$. In general, a basis for $\mathbf{R}^n$ is

$$\{(1,0,0,0,\ldots,0),(0,1,0,0,\ldots,0),(0,0,1,0,\ldots,0),$$
$$\ldots,(0,0,0,0,\ldots,0,1)\}.$$

Note. The set of all sequences

$$\{(x_1, x_2, \ldots) \mid x_i \in \mathbf{R}\}$$

forms an infinite dimensional vector space with basis

$$\{(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, 0, \ldots)\}.$$

(...we have not defined the term "basis" for an infinite dimensional vector space, though!)

Note. It is tempting to think of the collection of basis vectors as pointing in a bunch of different *directions*. Well... **give into the dark side!** We still need an idea of orthogonality to complete this geometric interpretation, but that's where we're headed! It may seem wierd to think of "the direction x^2" or the "direction $\cos \omega t$," BUT DO IT!

LOT'S OF GEOMETRY TO FOLLOW!

1.4 VECTOR SPACES - ISOMORPHISM

Note/Definition. An *isomorphism* between vector spaces V_1 and V_2, both over the scalar field F, is a function π which maps the vectors of V_1 to the vectors of V_2 such that the operations of vector addition and scalar multiplication are preserved. We say V_1 is *isomorphic* to V_2, denoted $V_1 \cong V_2$. For example, $\mathbf{R}^3 \cong \mathcal{P}_2$. An isomorphism between $\mathbf{R}^3$ and $\mathcal{P}_2$ is the mapping $\pi : \mathbf{R}^3 \to \mathcal{P}_2$ defined as $\pi((a_0, a_1, a_2)) = a_0 + a_1 x + a_2 x^2$. Notice that

1. vector addition is preserved:

$$\begin{aligned}
\pi((a_0, a_1, a_2) + (b_0, b_1, b_2)) &= \pi((a_0 + b_0, a_1 + b_1, a_2 + b_2)) \\
&= (a_0 + b_0) + (a_1 + b_1)x \\
&\quad + (a_2 + b_2)x^2 \\
&= (a_0 + a_1 x + a_2 x^2) \\
&\quad + (b_0 + b_1 x + b_2 x^2) \\
&= \pi((a_0, a_1, a_2)) + \pi((b_0, b_1, b_2))
\end{aligned}$$

2. scalar multiplication is preserved:

$$\begin{aligned}\pi(\alpha(a_0, a_1, a_2)) &= \pi((\alpha a_0, \alpha a_1, \alpha a_2)) \\ &= \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 \\ &= \alpha(a_0 + a_1 x + a_2 x^2) \\ &= \alpha\pi(a_0, a_1, a_2).\end{aligned}$$

Theorem. "Fundamental Theorem of Linear Algebra"
An n dimensional vector space over the field $\mathbf{R}$ (or F in general) is isomorphic to $\mathbf{R}^n$ (or F^n in general).

Note. This theorem tells us that all finite dimensional vector spaces "look like" $\mathbf{R}^n$. So... what does an infinite dimensional vector space "look like?"