BUTT-HEAD Seminar

II. Vector Spaces and Hilbert Spaces: Norms, Completeness, Inner Products and Hilbert Spaces

(From "Introduction to Hilbert Spaces with Applications" by L. Debnath and R. Mikusiński.)

Robert Gardner

East Tennessee State University
Department of Mathematics
Institute of Mathematical and Physical Sciences

Spring 1998

2.1 VECTOR SPACES AND HILBERT SPACES - NORMS

Definition. A real function $||\cdot||$ on a vector space H is a *norm* if

(a) $||x|| \geq 0$ for all $x \in V$ and $||x|| = 0$ if and only if $x = 0$

(b) $||\lambda x|| = |\lambda|||x||$ for all $x \in V$ and $\lambda \in F$

(c) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in V$ (triangle inequality).

Note. If $||\cdot||$ is a norm on a vector space, then $d(x, y) = ||x - y||$ defines a *metric* on the vector space with which we can measure distance.

Example. A norm on $\mathbf{R}^n$ is

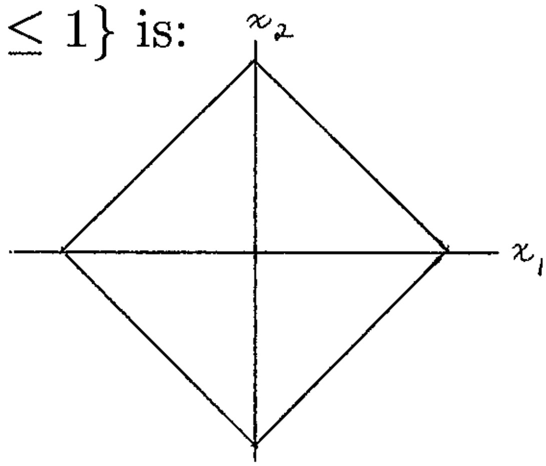
$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$. This is the *Euclidean norm* and can be used to define the *Euclidean metric* on $\mathbf{R}^n$. Notice that for $n = 1$ this is simply absolute value.

Example. Another norm on $\mathbf{R}^n$ is

$$\|x\| = |x_1| + |x_2| + \cdots + |x_n|$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$. For $n = 2$ the unit "ball" $\{x \mid \|x\| \leq 1\}$ is:



Example. A norm on the vector space of all functions continuous on $[0, 1]$ is

$$\|f\| = \max_{x \in [0,1]} |f(x)|.$$

Another norm on this space is

$$\|f\| = \left\{\int_0^1 |f(x)|^2 \, dx\right\}^{1/2}.$$

Example. A norm on $\mathbf{C}^n$ is

$$\|z\| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$

where $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$.

2.2 VECTOR SPACES AND HILBERT SPACES - COMPLETENESS

Note. We now need to explore the difficult subject of completeness.

Definition. A sequence of vectors $\{x_n\}$ in a normed space is a *Cauchy sequence* if for every $\epsilon > 0$, there exists a number M such that $\|x_m - x_n\| < \epsilon$ for all $m, n > M$.

Note. A sequence of real numbers is Cauchy if and only if it is convergent.

Definition. A vector space is *complete* if every Cauchy sequence converges.

Geometric Note. When you hear the term "complete," think "no holes." The rational numbers

$$\mathbf{Q} = \{p/q \mid p, q \in \mathbf{Z}, q \neq 0\}$$

is not a complete vector space (here we take the scalar field to be **Q** itself) since the sequence

$$\{1, 1.4, 1.41, 1.414, \ldots\}$$

is Cauchy but does not converge *in this space* (since the limit is $\sqrt{2}$). In some sense, **Q** is not complete since it has holes! In particular, it has a hole at $\sqrt{2}$.

Definition. A complete normed vector space is a *Banach space.*

Note. The real numbers are complete (in fact, this is part of the definiton of **R**) and so form a Banach space. More generally, $\mathbf{R}^n$ and $\mathbf{C}^n$ form Banach spaces.

Example. The vector space of all square summable sequences of complex numbers

$$l^2 = \left\{ (z_1, z_2, \ldots) \mid z_i \in \mathbf{C} \text{ and } \sum_{i=1}^{\infty} |z_i|^2 < \infty \right\}$$

with the norm

$$\|(z_1, z_2, \ldots)\| = \left(\sum_{i=1}^{\infty} |z_i|^2 \right)^{1/2}$$

is a (very fundamental) Banach space. This is a somewhat difficult result and it is not even clear that this space is closed under addition.

2.3 VECTOR SPACES AND HILBERT SPACES - INNER PRODUCTS

Note. We are ultimately interested in generalizing the idea of dot product in $\mathbf{R}^n$ (or $\mathbf{C}^n$) to the setting of infinite dimensional spaces.

Definition. Let V be a vector space over the field of scalars $\mathbf{C}$. A mapping

$$(\cdot, \cdot) : V \times V \to \mathbf{C}$$

is an *inner product* in V if for any $x, y, z \in V$ and $\alpha, \beta \in \mathbf{C}$, the following hold:
(a) $(x, y) = \overline{(y, x)}$ (the bar represents complex conjugate),
(b) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$,
(c) $(x, x) \geq 0$ and $(x, x) = 0$ implies $x = 0$.
A vector space with an inner product is an *inner product space* (or *pre-Hilbert space*).

Example. An inner product can be put on the real vector space $\mathbf{R}^n$ as follows: for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, define

$$(x, y) = x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

Example. An inner product can be put on the complex vector space $\mathbf{C}^n$ as follows: for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, define

$$(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Example. An inner product can be put on the vector space l^2 as follows: for $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ define

$$(x, y) = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

Notice. In each of the three inner product spaces above, the inner product can be used to define a norm: $\|x\| = \sqrt{(x, x)}$. In fact, in each case the norm determined by the inner product is the norm on the vector space we mentioned when these spaces were originally introduced.

Example. The space $L^2([a,b])$ of all square (Lebesgue) integrable functions on the real interval $[a,b]$:

$$L^2([a,b]) = \left\{ f : [a,b] \to \mathbf{C} \;\middle|\; \int_a^b |f(x)|^2 \, dx < \infty \right\}$$

has as an inner product defined by

$$(f,g) = \int_a^b f(x)\overline{g(x)}\, dx.$$

THIS IS AN IMPORTANT INNER PRODUCT SPACE!

Definition/Theorem. An inner product space has a *norm* $||\cdot||$ induced by the inner product as follows: $||x|| = \sqrt{(x,x)}$.

Note. From the definition of inner product, it is clear that $||x|| \geq 0$ and $||\lambda x|| = |\lambda|\,||x||$ for all scalars λ and vectors x. To establish the triangle inequality is a bit harder.

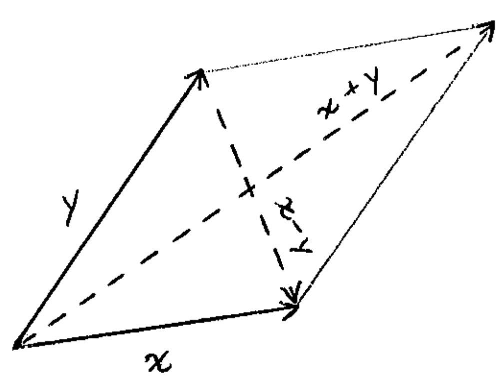
Note. Since an inner product space necessarily has a norm, it is of interest to know if this normed space is a Banach space (i.e. if it is complete).

Note. A geometrically suggestive result is the following:

Theorem (Parallelogram Law).
For any two elements x and y of an inner product space, we have

$$||x+y||^2 + ||x-y||^2 = 4(||x||^2 + ||y||^2).$$

In $\mathbf{R}^2$, this implies:



Definition. Two vectors x and y in an inner product space are *orthogonal* if $(x, y) = 0$.

Note. Another geometrically suggestive result is:

Theorem (Pythagorean Formula.)
If x and y are orthogonal vectors in an inner product space, then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

In $\mathbf{R}^2$, this is simply the Pythagorean Theorem.

2.4 VECTOR SPACES AND HILBERT SPACES - HILBERT SPACES

Definition. A complete inner product space is a *Hilbert space.*

Note. We have the following general inclusions:

Hilbert spaces $\subset$ Banach spaces $\subset$ vector spaces

Example. We have already seen several examples of Hilbert spaces. Some of these are:

(a) $\mathbf{C}^n = \{(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbf{C}\}$.

(b) $l^2 = \left\{(z_1, z_2, \ldots) \mid z_i \in \mathbf{C} \text{ and } \sum_{i=1}^{\infty} |z_i|^2 < \infty\right\}$.

(c) $L^2([a,b]) = \left\{f : [a,b] \to \mathbf{C} \;\middle|\; \int_a^b |f(x)|^2\,dx < \infty\right\}$.

To establish that these are in fact Hilbert spaces, the only difficult part is the establishment of completeness.

2.4.1 Bases in Hilbert Spaces

Note. We now explore the idea of a basis of a Hilbert space (the following two definitions are from "An Introduction to Nonharmonic Fourier Series" by R. Young).

Definition. A *Hamel basis* of an infinite dimensional Banach space is a linearly independent set that spans the space (this is the same as the definition of basis in finite dimensional vector spaces).

Note. If we have a Hamel basis H of a Banach space B then each $x \in B$ can be written as $x = \sum_{i=1}^{n} \alpha_i h_i$ for some FINITE collection $\{h_i\} \subset H$ and for some set of scalars $\{\alpha_i\}$. Unfortunately, such bases cannot in general be constructed (and therefore are of limited use). In fact, the proof of the existence of a Hamel basis for an arbitrary Banach space requires the use of the Axiom of Choice.

Definition. A *Schauder basis* of an infinite dimensional Banach space is a set of vectors $\{x_1, x_2, \ldots\}$ such that for any vector x in the Banach space, there is a unique sequence of scalars $\{\alpha_1, \alpha_2, \ldots\}$ such that $x = \sum_{i=1}^{\infty} \alpha_i x_i$.

Note. Not every Banach space has a Schauder basis. We are interested in Hilbert spaces which have Schauder bases.

2.4.2 Seperable Hilbert Spaces

Note. We now need a few "mathy" definitions.

Definition. A set is *countable* if a complete "listing" of the set can be made.

Examples.

The natural numbers are countable: $\{1, 2, 3, \ldots\}$.

The integers are countable: $\{0, 1, -1, 2, -2, \ldots\}$.

Surprisingly, the rational numbers are countable (even though they are very different from the integers *topologically*).

The real numbers are not countable!

Definition. Suppose X is a normed space. A set D is *dense* in X if every open set in X includes an element of D.

Example. The rational numbers are dense in the real numbers. The integers are not dense in the reals.

Definition. A Hilbert space with a countable dense subset is *seperable.*

Note. Since $\mathbf{Q}$ is countable and dense in $\mathbf{R}$, then $\mathbf{R}$ forms a seperable Hilbert space (in fact, any finite dimensional Hilbert space is seperable - and remember, a finite dimensional Hilbert space/vector space is isomorphic to either $\mathbf{R}^n$ or $\mathbf{C}^n$ depending on the scalar field).

Definition. A subset X of a Hilbert space is an *orthonormal set* if $\|x\| = 1$ for all $x \in X$ and $(x, y) = 0$ (that is, x and y are orthogonal) for all $x, y \in X$.

Note. As in $\mathbf{R}^n$, a "nice" basis for a Hilbert space would be orthonormal.

Theorem. A Hilbert space is seperable if and only if it has an orthonormal Schauder basis. (From now on, when we say "basis" it is understood that we mean "Schauder basis.")

Definition. A Hilbert space H_1 is *isomorphic* to a Hilbert space H_2 if there exists a one-to-one linear mapping T from H_1 onto H_2 such that $(T(x), T(y)) = (x, y)$ for every $x, y \in H_1$.

Note. Now for the BIG RESULT! Recall that the "Fundamental Theorem of Linear Algebra" tells you what a finite dimensional vector space "looks like."

Theorem (Riesz-Fisher Theorem).
An infinite dimensional Hilbert space with scalar field $\mathbf{C}$ (that is, a seperable Hilbert space that is not isomorphic to some $\mathbf{C}^n$) is isomorphic to

$$l^2 = \left\{(z_1, z_2, \ldots) \mid z_i \in \mathbf{C} \text{ and } \sum_{i=1}^{\infty} |z_i|^2 < \infty\right\}.$$

Note. An orthonormal basis for l^2 is

$$\{e_1, e_2, e_3, \ldots\} = \{(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots),$$
$$(0, 0, 1, 0, \ldots), \ldots\}.$$