# BUTT-HEAD Seminar

# III. Linear Operators on Hilbert Spaces: Operators, Norms, Self Adjoint Operators

(From "Introduction to Hilbert Spaces with Applications"
by L. Debnath and R. Mikusiński.)

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# 3.1 LINEAR OPERATORS ON HILBERT SPACES - LINEAR OPERATORS AND OPERATOR NORMS

**Definition.** An *operator* is a mapping from one vector space to another (not necessarily different) vector space.

**Definition.** A *linear operator* $L$ which maps one vector space to another (where the vector spaces have the same scalar field) is an operator satisfying

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$$

for all scalars $\alpha$ and $\beta$ and for all vectors $x$ and $y$.

**Note.** We will see that in the quantum theory, observables (such as position, momentum and energy) are represented by a certain type of linear operator.

**Theorem.** Every linear operator on $\mathbf{R}^n$ (or $\mathbf{C}^n$) is represented by an $n \times n$ real (or complex) matrix.

**Proof.** Let $A$ be a linear operator and $\mathbf{C}^n$ have "standard" orthonormal basis $\{e_1, e_2, \ldots, e_n\}$. Then for $x = \sum_{j=1}^{n} a_j e_j \in \mathbf{C}^n$, we have

$$Ax = A\left(\sum_{j=1}^{n} a_j e_j\right) = \sum_{j=1}^{n} a_j A(e_j).$$

Therefore

$$(Ax, e_i) = \left(\sum_{j=1}^{n} a_j A e_j, e_i\right) = \sum_{j=1}^{n} a_j (Ae_j, e_i) = \sum_{j=1}^{n} a_j a_{i,j}$$

where $a_{i,j} = (Ae_j, e_i)$. Therefore $A$ can be represented by the $n \times n$ matrix $(a_{i,j})$. ∎

**Example.** Consider the vector space

$$C_1([a,b]) = \{f : [a,b] \to \mathbf{C} \mid f' \text{ is continuous}\}.$$

Then $C_1([a,b])$ is a subspace of $L^2([a,b])$ and the operator $D$ defined as

$$D(f(t)) = \frac{d}{dt}[f(t)] = f'(t)$$

is a linear operator (in fact, $D : C_1([a,b]) \to C_0([a,b])$ where $C_0([a,b])$ is the vector space of all continuous functions on $[a,b]$). $D$ is called the *differential operator.*

**Example.** Let $z \in C_0([a,b])$ and define operator $A$ on $L^2([a,b])$ by

$$A(x(t)) = z(t)x(t).$$

$A$ is called a *multiplication operator.*

**Definition.** An operator on a vector space $V$ is *bounded* if there exists a nonnegative real $K$ such that $||Ax|| \leq K||x||$ for all $x \in V$. The *norm* of a bounded linear operator is the "smallest" such value $K$, or equivalently $||A|| = \sup_{||x||=1} ||Ax||$.

**Example.** The identity operator $\mathcal{I}$ on a vector space $V$ is defined as $\mathcal{I}(x) = x$ for all $x \in V$. Notice that $\mathcal{I}$ is bounded and $\|\mathcal{I}\| = 1$.

**Note.** A multiplier operator $A$ is bounded:

~~$\|A\|^2 \leq \sup_{\|x\|=1}$~~ $\|Ax\|^2 = \int_a^b |A(x(t))|^2 dt = \int_a^b |z(t)x(t)|^2 dt$

$$= \int_a^b |z(t)|^2 |x(t)|^2 dt \leq \max_{t \in [a,b]} |z(t)|^2 \int_a^b |x(t)|^2 dt$$

$$= \left( \max_{t \in [a,b]} |z(t)|^2 \right) \|x\|^2.$$

Notice that if $z(t) = K \in \mathbf{C}$ then $\|A\| = |K|$.

**Note.** The differential operator $D$ is not bounded. Consider the sequence of functions $f_n(t) = \sin(nt)$ for $n = 1, 2, 3, \ldots$, as elements of $L^2([-\pi, \pi])$. Then

$$\|f_n\| = \sqrt{\int_{-\pi}^{\pi} (\sin(nt))^2 dt} = \sqrt{\pi}$$

and

$$\|D(f_n)\| = \sqrt{\int_{-\pi}^{\pi} (n\cos(nt))^2 dt} = n\sqrt{\pi}.$$

Therefore $\|D(f_n)\| = n\|f_n\|$ and we see that $\sup\limits_{\|x\|=1} \|D(x)\|$ can be made arbitrarily large by taking $x_n = \dfrac{f_n}{\sqrt{\pi}} = \dfrac{\sin(nt)}{\sqrt{\pi}}$ (then $\|x_n\| = 1$ and $\|D(x_n)\| = n$, therefore $\sup\limits_{\|x_n\|=1} \|D(x)\| = \infty$).

**Definition.** The *product* of operators $A$ and $B$ on vector space $V$ is defined as $AB(x) = A(Bx)$ for all $x \in V$. If $AB = BA$ then $A$ and $B$ are *commuting operators.*

**Example.** The differential operator $D = \dfrac{d}{dt}$ and the multiplier operator $A(x(t)) = tx(t)$ do not commute.

**Example.** The identity operator commutes with all operators.

**Theorem.** The product $AB$ of bounded linear operators $A$ and $B$ is a bounded linear operator and $\|AB\| \leq \|A\| \, \|B\|$.

**Theorem.** A bounded linear operator on a seperable infinite dimensional Hilbert space can be represented by an infinite matrix.

**Example.** The operator $Ax = \alpha x$ where $\alpha$ is a fixed scalar and $x \in l^2$ is represented by the infinite matrix

$$\begin{bmatrix} \alpha & 0 & 0 & \dots \\ 0 & \alpha & 0 & \dots \\ 0 & 0 & \alpha & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

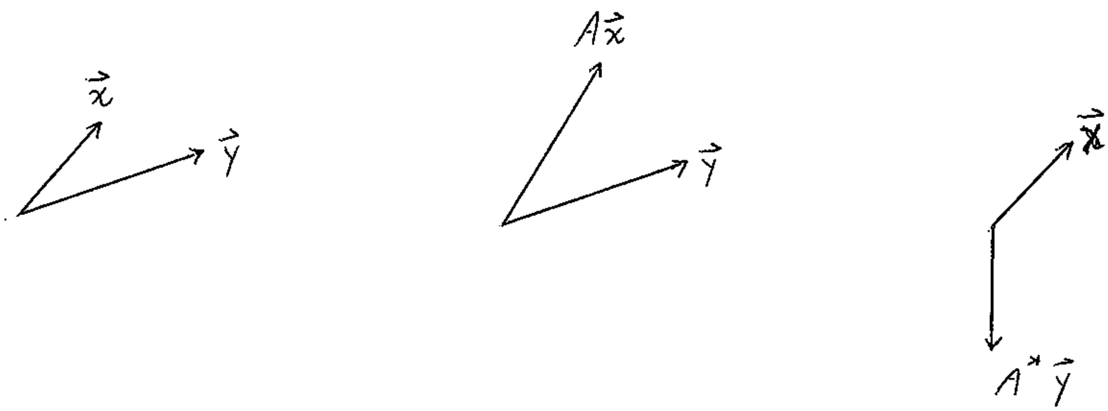
# 3.2 LINEAR OPERATORS ON HILBERT SPACES - ADJOINT AND SELF-ADJOINT OPERATORS

**Definition.** Let $A$ be a bounded linear operator on a Hilbert space $H$. The operator $A^* : H \to H$ defined by

$$(Ax, y) = (x, A^*y) \text{ for all } x, y \in H$$

is the *adjoint operator* of $A$.

**Note.** In $\mathbf{R}^2$, this implies:




$$Ax \cdot y = x \cdot A^*y$$

**Theorem.** Properties of Adjoint.

**1.** $(A + B)^* = A^* + B^*$

**2.** $(\alpha A)^* = \overline{\alpha} A^*$

**3.** $(A^*)^* = A$

**4.** $\mathcal{I}^* = \mathcal{I}$

**5.** $(AB) = B^* A^*$

**Theorem.** Suppose $A$ is a bounded linear operator. Then $A^*$ is bounded and $||A|| = ||A^*||$ and $||A^*A|| = ||A||^2$. (Notice that in general $||AB|| \leq ||A|| ||B||$.)

**Definition.** If $A = A^*$ then $A$ is a *self adjoint* (or *Hermetian*) operator.

**Example.** Let $A$ be the operator on $L^2([a,b])$ defined by

$$A(x(t)) = tx(t).$$

Then $A$ is self-adjoint since

$$(Ax, y) = \int_a^b tx(t)\overline{y(t)}\,dt = \int_a^b x(t)\overline{ty(t)}\,dt = (x, Ay).$$

**Example/Theorem.** If $A$ is an operator on $\mathbf{C}^n$, then $A$ is represented by an $n \times n$ matrix $A = (a_{i,j})$ (as above) and $A^*$ is represented by the matrix $A^* = (\overline{a_{j,i}})$.

**Proof.** Let $\{e_1, e_2, \ldots, e_n\}$ be the standard orthonormal basis for $\mathbf{C}^n$ (i.e., $e_i$ has $i$th entry 1 and all other entries 0). Suppose $A^*$ is represented by $(b_{i,j})$. Then we have

$$a_{i,j} = (Ae_j, e_i) = (e_j, A^*e_i) = \overline{(A^*e_j, e_i)} = \overline{b_{j,i}}.$$

Therefore $A^* = (b_{i,j}) = (\overline{a_{j,i}})$. ∎

**Note.** If $A$ is a self-adjoint operator on $\mathbf{C}^n$ and $A$ is represented by the $n \times n$ matrix $A = (a_{i,j})$, then $A^*$ is represented by an $n \times n$ matrix, the $(i, j)$ entry of which is $\overline{a_{j,i}}$. So $a_{i,j} = \overline{a_{j,i}}$ and $A$ equals its "conjugate transpose."

**Note.** The previous result holds in separable Hilbert spaces as well. That is, if $A$ is a bounded self-adjoint operator on a serparable (infinite dimensional) Hilbert space and $A$ is represented by the infinite matrix $(a_{i,j})$, then $A^*$ is represented by $(\overline{a_{j,i}})$ and we have $a_{i,j} = \overline{a_{j,i}}$.

**Theorem.** Let $A$ be a bounded operator on a Hilbert space. Then $A^*A$ and $A + A^*$ are self-adjoint.

**Theorem.** The product of two self-adjoint operators is self-adjoint if and only if the operators commute.

**Proof.** Let $A$ and $B$ be self-adjoint. Then

$$(ABx, y) = (Bx, A^*y) = (x, B^*A^*y) = (x, BAy).$$

So if $AB = BA$ then $AB$ is self-adjoint. Conversely, if $AB$ is self-adjoint, then $(AB)^* = AB$ and from the above, $(AB)^* = BA$, therefore $AB = BA$. ∎

# 3.3 LINEAR OPERATORS ON HILBERT SPACES - INVERSES AND UNITARY OPERATORS

**Definition.** Let $A$ be an operator with range $\mathcal{R}(A)$. An operator $B$ is the *inverse* of $A$ if $ABx = x$ for all $x \in \mathcal{R}(A)$ and $BAx = x$ for all $x \in \mathcal{D}(A)$ where $\mathcal{D}(A)$ is the domain of $A$. Operator $A$ is said to be invertible and $B$ is denoted as $A^{-1}$.

**Some Properties of $A^{-1}$**

**1.** If $A$ is linear, then $A^{-1}$ is linear.

**2.** $A$ is invertible if and only if $Ax = 0$ implies $x = 0$.

**3.** If $A$ is invertible and $x_1, x_2, \ldots, x_n$ are linearly independent, then $Ax_1, Ax_2, \ldots, Ax_n$ are linearly independent.

**4.** If $A$ and $B$ are invertible, then $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

**Definition.** A bounded operator $T$ on a Hilbert space is *unitary* if $T^*T = TT^* = \mathcal{I}$. That is, $T$ is unitary if and only if $T^{-1} = T^*$.

**Example.** Define $T$ on $L^2([0,1])$ as

$$T(x(t)) = x(1-t).$$

Then

$$T(T(x(t))) = T(x(1-t)) = x(1-(1-t)) = x(t)$$

and $T = T^{-1}$. Also

$$\begin{aligned}(Tx, y) &= \int_0^1 x(1-t)y(t)\,dt = -\int_1^0 x(u)y(1-u)\,du \\ &= \int_0^1 x(u)y(1-u)\,du = (x, Ty)\end{aligned}$$

and so $T = T^*$. Therefore, $T^* = T^{-1}$ and $T$ is unitary.

**Definition.** Let $S$ be a non-empty subspace of a Hilbert space $H$. An element $x \in H$ is *orthogonal* to $S$, denoted $x \perp S$, if $(x, y) = 0$ for every $y \in S$. The set of all elements of $H$ orthogonal to $S$, denoted $S^\perp$, is the *orthogonal complement* of $S$.

**Example.** In $\mathbf{C}^3$, with $S = \{(z, 0, 0) \mid z \in \mathbf{C}\}$ (clearly a subspace), $S^\perp = \{(0, z_1, z_2) \mid z_1, z_2 \in \mathbf{C}\}$.

**Definition.** Let $S$ be (topologically) closed subspace of a Hilbert space $H$. The operator $P$ defined as

$$P(x) = y \text{ for } x = y + z, y \in S \text{ and } z \in S^\perp$$

is the *projection operator* onto $S$. The vector $y$ is the *projection of $x$ onto $S$*.

**Example.** Let $S$ be a closed subspace of a Hilbert space $H$ and let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of $S$. Then the projection operator $P$ onto $S$ is defined by

$$P(x) = \sum_{n=1}^{\infty} (x, e_n)e_n.$$

In particular, if $S$ is of dimension 1 (of special interest in quantum mechanics), then for $v \in S$ where $||v|| = 1$, we have $P(x) = (x, v)v$.

# 3.4 LINEAR OPERATORS ON HILBERT SPACES - EIGENVALUES AND EIGENVECTORS

**Definition.** Let $A$ be an operator on a complex vector space $E$. A complex number $\lambda$ is an *eigenvalue* of $A$ if there is a non-zero vector $u \in E$ such that $Au = \lambda u$. A vector satisfying this condition is an *eigenvector* of $A$ corresponding to eigenvalue $\lambda$ (in a function space, an eigenvector is also called an *eigenfunction*).

**Definition.** Let $A$ be an operator on a normed space $E$. The operator

$$A_\lambda = (A - \lambda \mathcal{I})^{-1}$$

is called the *resolvent* of $A$. The values of $\lambda$ for which $A_\lambda$ is defined on the whole space $E$ and is bounded are called the *regular points* of $A$. The set of all $\lambda$s which are not regular is called the *spectrum* of $A$.

The set of all eigenvalues (which are a subset of the spectrum) is called the *point spectrum.* The remaining part of the spectrum (that is, that set of all $\lambda$s for which $A_\lambda$ exists but is unbounded) is called the *continuous spectrum.*

**Theorem.** The collection of all eigenvalues corresponding to one particular eigenvalue of an operator is a vector space.

**Definition.** The set of all eigenvectors corresponding to one particular eigenvalue $\lambda$ is called the *eigenspace* of $\lambda$. The dimension of that space is called the *multiplicity* of $\lambda$ (this is consistent with the finite dimensional idea where the multiplicity of an eigenvalue of a matrix corresponds to its multiplicity as a zero of the equation $A - \lambda\mathcal{I} = 0$).

**Theorem.** All eigenvalues of a self-adjoint operator on a Hilbert space are real.

**Proof.** Let $\lambda$ be an eigenvalue of a self-adjoint operator $A$, and let $u$ be an eigenvector of $\lambda$, $u \neq 0$. Then

$$\begin{aligned}\lambda(u,u) &= (\lambda u, u) = (Au, u) \\ &= (u, Au) = (u, \lambda u) = \overline{\lambda}(u,u).\end{aligned}$$

Since $(u,u) \geq 0$, we have $\lambda = \overline{\lambda}$, and therefore, $\lambda$ is real. ∎

**Theorem.** Eigenvectors corresponding to distinct eigenvalues of a self-adjoint or unitary operator on a Hilbert space are orthogonal.

**Definition.** An operator $A$ in a Hilbert space $H$ is *compact* (or sometimes called *completely continuous*) if for every bounded sequence $\{x_n\}$ in $H$, the sequence $\{Ax_n\}$ contains a convergent subsequence.

**Note.** Every compact operator is bounded, but not every bounded operator is compact (the identity operator $\mathcal{I}$ is bounded, but not compact... consider the standard orthonormal basis in $l^2$).

**Theorem. The Spectral Theorem for Self-Adjoint Compact Operators.**
Let $A$ be a self-adjoint compact operator on an infinite dimensional Hilbert space $H$. Then there exists an orthonormal basis of $H$, $\{v_n\}$, consisting of eigenvectors of $A$. Moreover, for every $x \in H$,

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x, v_n) v_n,$$

where $\lambda_n$ is the eigenvalue corresponding to $v_n$.

**Theorem.** For any two commuting self-adjoint compact operators $A$ and $B$ on a Hilbert space $H$, there exists an orthonormal basis of $H$ consisting of vectors which are eigenvalues of both $A$ and $B$.

NOW THAT WE HAVE DEVELOPED THE MATHEMATICAL BACKGROUND, WE ARE READY FOR THE <u>REAL</u> REASON WE ARE ALL HERE...

THE APPLICATIONS OF THIS STUFF TO QUANTUM MECHANICS!!!

[Gardner exit stage right. Shanks enter stage left.]

# APPENDIX A - Unitary Operators

**Recall.** A bounded operator $T$ is *unitary* if and only if $T$ is invertible and $T^{-1} = T^*$.

**Definition.** A bounded operator $T$ is *isometric* if it preserves lengths: $\|Tx\| = \|x\|$ for all $x$.

**Theorem.** A unitary operator is isometric.

**Proof.**

$$\|Tx\| = (Tx, Tx) = (x, T^*Tx)$$

$$= (x, T^{-1}Tx) = (x, x) = \|x\|.$$

∎

**Theorem.** A unitary operator "preserves angles." That is, $(x, y) = (Tx, Ty)$.

**Proof.**

$$(x, y) = (x, \mathcal{I}y) = (x, T^*Ty) = (Tx, Ty).$$

∎