Dr. Bob's Favorite Results on Complex Polynomials

Robert "Dr. Bob" Gardner

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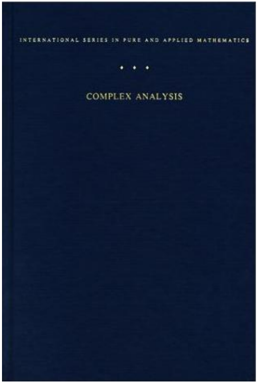
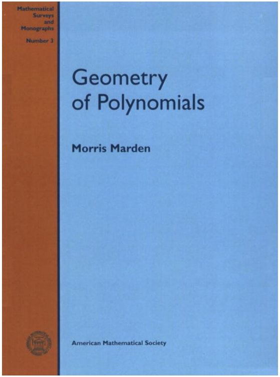
Primary Sources

Primary Sources Used in this Presentation



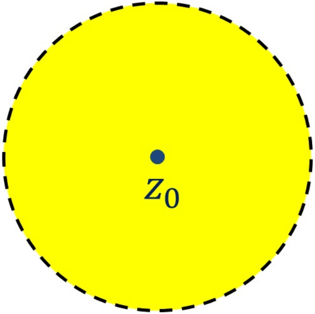
Lars Ahlfors, *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*,



Morris Marden, *Geometry of Polynomials*, Mathematical Monographs and Surveys #3, AMS: 1986.

Liouville's Theorem and the Maximum Modulus Theorem

Recall. A function is *analytic* if it has a power series representation. A function of a complex variable, $f(z)$, is analytic (and therefore has a power series representation) at point z_0 if f is continuously differentiable at z_0.



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Theorem

If f is analytic in a neighborhood of z_0 and C is a positively oriented simple closed curve in the neighborhood with z_0 as an interior point, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}.$$

Augustin Cauchy (1789–1857)

Note. If we let C be a circle of radius R with center z_0, then Cauchy's Formula allows us to put a bound on derivatives of f:

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \right|$$

$$\leq \left| \frac{n!}{2\pi i} \right| \int_C \left| \frac{f(z)\,dz}{(z-z_0)^{n+1}} \right| \leq \frac{n!}{2\pi |i|} \int_C \frac{|f(z)|\,|dz|}{|z-z_0|^{n+1}}$$

$$\leq \frac{n!}{2\pi} \frac{M_R 2\pi R}{R^{n+1}} = \frac{n! M_R}{R^n},$$

where M_R is an upper bound of $|f(z)|$ over C. This is called *Cauchy's Inequality*.

Theorem

Liouville's Theorem. *If f is a function analytic in the entire complex plane (f is called an entire function) which is bounded in modulus, then f is a constant function.*

Proof. Let M be a bound on $|f(z)|$. Then by Cauchy's Inequality with $n = 1$, $|f'(z_0)| \leq M/R$. This is true for any z_0 and for any R since f is entire. Since we can let $R \to \infty$ then we see that $|f'(z_0)| = 0$ and f must be constant. □

Joseph Liouville (1809–1882)

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Question. The only bounded entire functions of a complex variable are constant functions. Is this the case for functions of a real variable?

Answer. NO! Consider $f(x) = \sin x$ and $g(x) = e^{-x^2}$.

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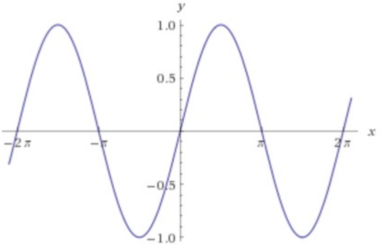

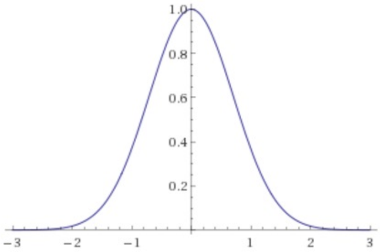

From Wolfram Alpha.

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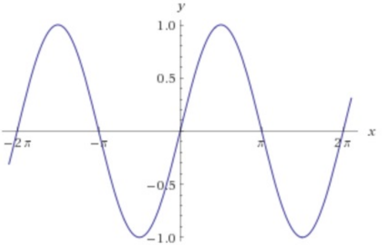

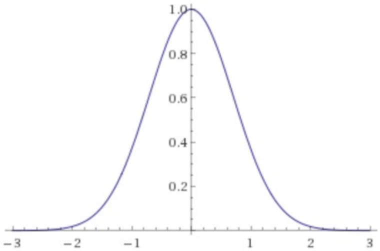

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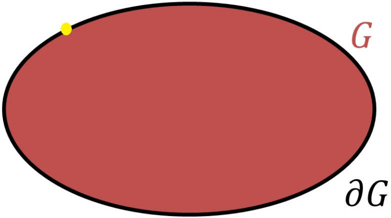
The Maximum Modulus Theorem

Theorem

The Maximum Modulus Theorem. *Let G be a bounded open set in $\mathbb{C}$ and suppose f is continuous on the closure of G, $cl(H)$, and analytic in G. Then*

$$\max\{|f(z)| \mid z \in cl(G)\} = \max\{|f(z)| \mid z \in \partial G\}.$$

Also, if $\max\{|f(z)| \mid z \in G\} = \max\{|f(z)| \mid z \in \partial G\}$, *then f is constant.*



The Fundamental Theorem of Algebra

Carl Friedrich Gauss (1777–1855)

The Fundamental Theorem of Algebra

Theorem

The Fundamental Theorem of Algebra. *Any complex polynomial*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 \ (a_n \neq 0)$$

of degree n ($n \geq 1$) has at least one complex zero. That is, there exists at least one point z_0 such that $p(z_0) = 0$.

Note. It follows that p can be factored into n (not necessarily distinct) linear terms:

$$p(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_{n-1})(z - z_n),$$

where the zeros of p are $z_1, z_2, \ldots, z_n$.

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Proof. Suppose not. ASSUME p is never zero. Define the function $f(z) = 1/p(z)$. Then f is an entire function. Now $|p(z)| \to \infty$ as $|z| \to \infty$. Therefore $\frac{1}{p(z)} \to 0$ as $|z| \to \infty$. So for some R, $\frac{1}{|p(z)|} \leq 1$ for $|z| \geq R$. Now by the Extreme Value Theorem, for some M,

$$\left| \frac{1}{p(z)} \right| \leq M \text{ for } |z| \leq R.$$

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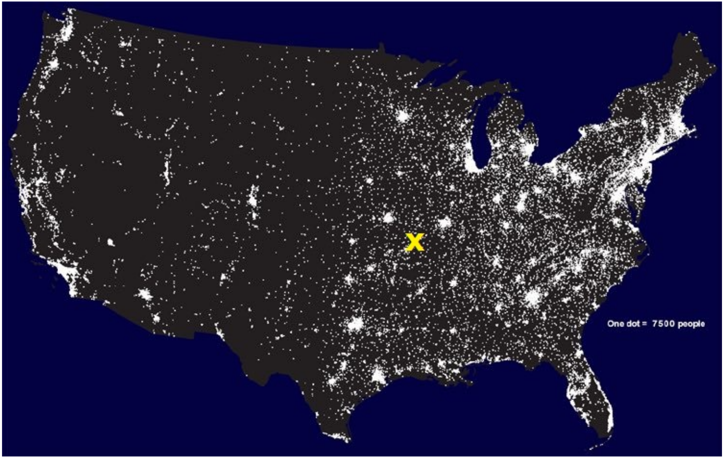
The Centroid Theorem



The Centroid Theorem

Theorem

The Centroid Theorem. *The centroid of the zeros of a polynomial is the same as the centroid of the zeros of the derivative of the polynomial.*

Proof. Let polynomial p have zeros $z_1, z_2, \ldots, z_n$. Then $p(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{k=1}^{n}(z - z_k)$. Multiplying out, we find that the coefficient of z^{n-1} is $a_{n-1} = -a_n(z_1 + z_2 + \cdots + z_n)$.

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$$\frac{z_1 + z_2 + \cdots + z_n}{n} = \left(\frac{1}{n}\right)\left(\frac{-a_{n-1}}{a_n}\right) = \frac{-a_{n-1}}{na_n}.$$

Let the zeros of p' be $w_1, w_2, \ldots, w_{n-1}$. Then

$$p'(z) = \sum_{k=1}^{n} k a_k z^{k-1} = na_n \prod_{k=1}^{n-1}(z - w_k).$$

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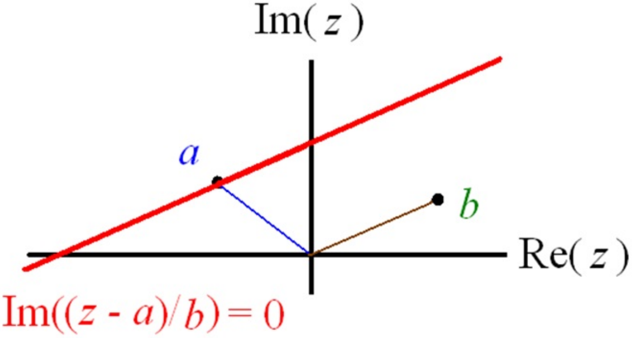
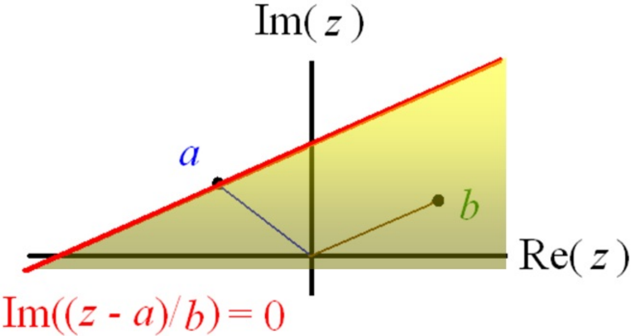
The Lucas Theorem

François A. E. Lucas (1842–1891)

Note. Recall that a line in the complex plane can be represented by an equation of the form $\text{Im}((z-a)/b) = 0$ where the line is "parallel" to the vector b and translated from the origin by an amount a (here we are knowingly blurring the distinction between vectors in $\mathbb{R}^2$ and numbers in $\mathbb{C}$).



Note. We can represent a closed half-plane with the equation $\text{Im}((z-a)/b) \leq 0$. The represents the half-plane to the right of the line $\text{Im}((z-a)/b) = 0$ when traveling along the line in the "direction" of b.



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The Lucas Theorem (1874). *If all the zeros of a polynomial p lie in a half-plane in the complex plane, then all the zeros of the derivative p' lie in the same half-plane.*

Proof. By the Fundamental Theorem of Algebra, we can factor p as $p(z) = a_n(z - r_1)(z - r_2)\cdots(z - r_n)$. So

$$\log p(z) = \log a_n + \log(z - r_1) + \log(z - r_2) + \cdots + \log(z - r_n)$$

and differentiating both sides gives

$$\frac{p'(z)}{p(z)} = \frac{1}{z - r_1} + \frac{1}{z - r_2} + \cdots + \frac{1}{z - r_n} = \sum_{k=1}^{n} \frac{1}{z - r_k}. \qquad (1)$$

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Proof (continued). Suppose the half-plane H that contains all the zeros of $p(z)$ is described by $\text{Im}((z-a)/b) \leq 0$. Then

$$\text{Im}((r_1 - a)/b) \leq 0, \, \text{Im}((r_2 - a)/b) \leq 0, \, \ldots, \, \text{Im}((r_n - a)/b) \leq 0.$$

Now let z^* be some number not in H. We want to show that $p'(z^*) \neq 0$ (this will mean that all the zeros of $p'(z)$ are in H). Well, $\text{Im}((z^* - a)/b) > 0$. Let r_k be some zero of p.

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$$\text{Im}\left(\frac{z^* - r_k}{b}\right) = \text{Im}\left(\frac{z^* - a - r_k + a}{b}\right) = \text{Im}\left(\frac{z^* - a}{b}\right) - \text{Im}\left(\frac{r_k - a}{b}\right) > 0.$$

(Notice that $\text{Im}((z^* - a)/b) > 0$ since z^* is not in H and $-\text{Im}((r_k - a)/b) \geq 0$ since r_k is in H.) The imaginary parts of reciprocal numbers have opposite signs, so $\text{Im}(b/(z^* - r_k)) < 0$.

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Proof (continued). Recall

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Applying (1),

$$\mathrm{Im}\left(\frac{bp'(z^*)}{p(z^*)}\right) = \sum_{k=1}^{n} \mathrm{Im}\left(\frac{b}{z^* - r_k}\right) < 0.$$

So $p'(z^*)/p(z^*) \neq 0$ and $p'(z^*) \neq 0$. Therefore if $p'(z) = 0$ then $z \in H$. □

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Corollary to The Lucas Theorem

Note. With repeated application of the Lucas Theorem, we can prove the following corollary.

Corollary

The convex polygon in the complex plane which contains all the zeros of a polynomial p also contains all the zeros of p'.

Note. For example, we might have zeros of p and its derivatives as follows. . .

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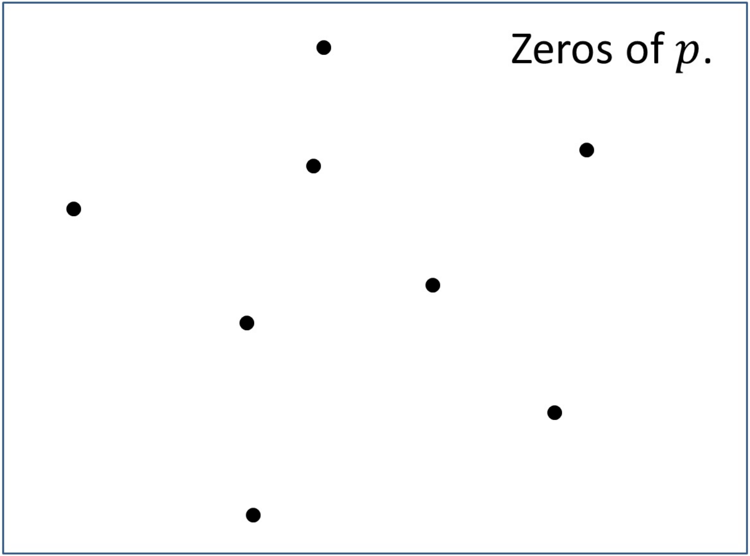
The Lucas Theorem, Pictures



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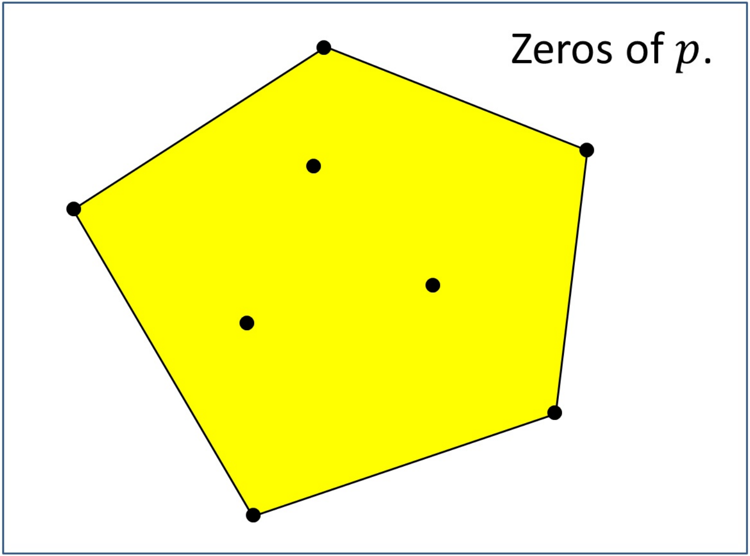

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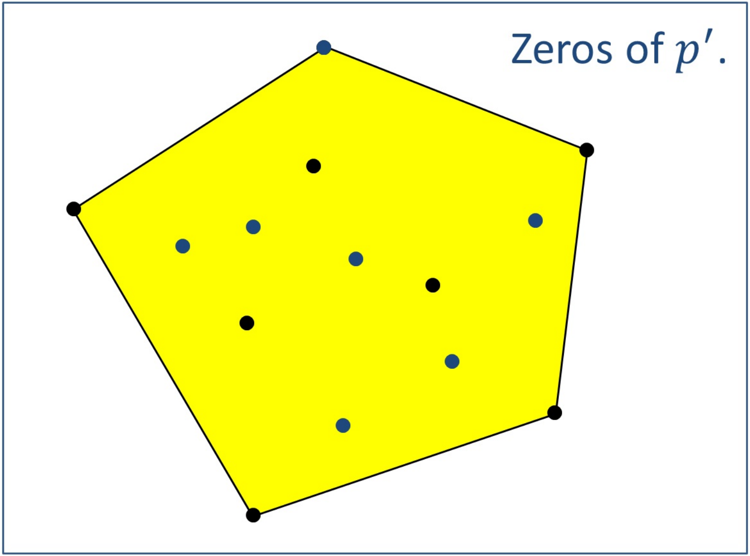

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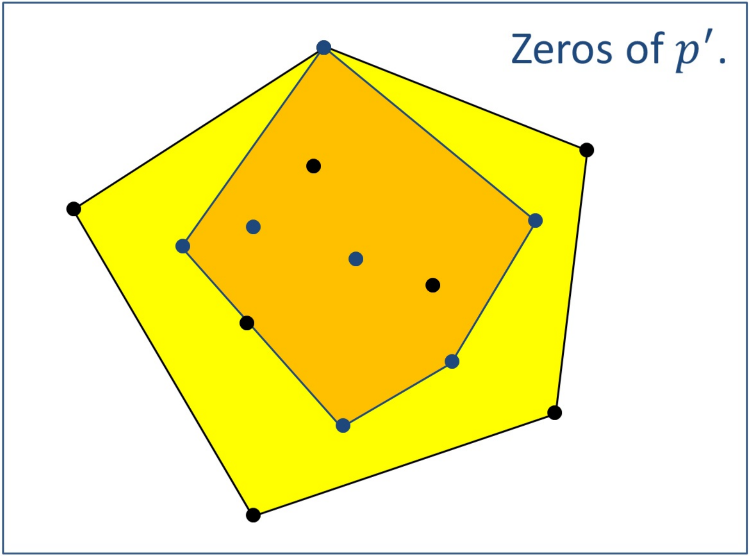

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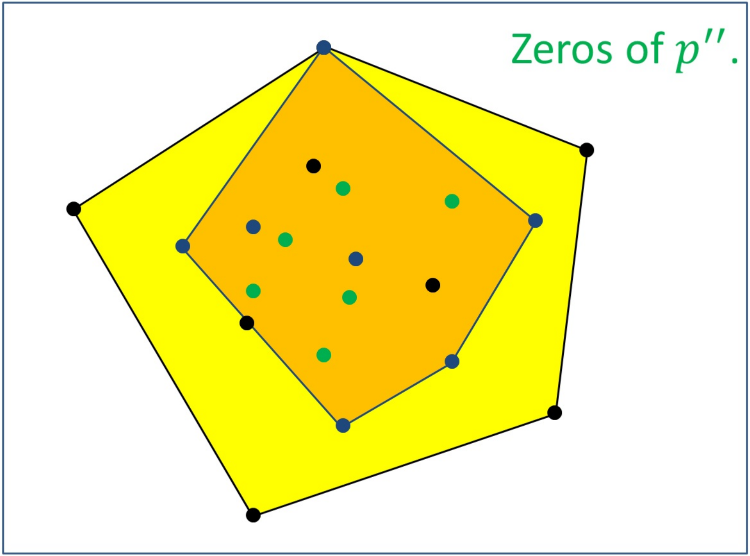

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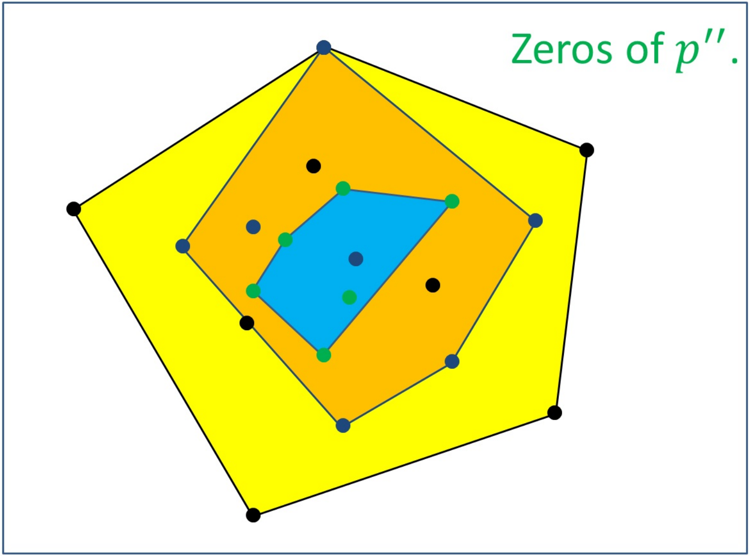
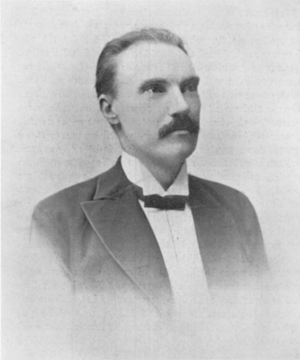


The Eneström-Kakeya Theorem



Gustav Eneström (1852–1923)

Sōichi Kakeya (1886–1947)

Eneström-Kakeya Theorem

Note. Gustav Eneström while studying the theory of pensions was the first to publish a result concerning the location of the zeros of a polynomial with monotone, real, nonnegative coefficients. He published his work in 1893 in Swedish. Sōichi Kakeya published a similar result in 1912 in English.

Theorem

Eneström-Kakeya Theorem. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n with coefficients satisfying* $0 \le a_0 \le a_1 \le \cdots \le a_n$, *then all the zeros of p lie in* $|z| \le 1$.

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Proof. Define f by the equation $p(z)(1-z) = a_0+(a_1-a_0)z+(a_2-a_1)z^2+\cdots+(a_n-a_{n-1})z^n-a_nz^{n+1} = f(z)-a_nz^{n+1}$. Then for $|z|=1$, we have

$$\begin{aligned} |f(z)| &\le |a_0|+|a_1-a_0|+|a_2-a_1|+\cdots+|a_n-a_{n-1}| \\ &= a_0+(a_1-a_0)+(a_2-a_1)+\cdots+(a_n-a_{n-1}) = a_n. \end{aligned}$$

Notice that the function $z^nf(1/z)=\sum_{j=0}^n(a_j-a_{j-1})z^{n-j}$ (where we take $a_{-1}=0$) has the same bound on $|z|=1$ as f. Namely, $|z^nf(1/z)|\le a_n$ for $|z|=1$. Since $z^nf(1/z)$ is analytic in $|z|\le 1$, we have $|z^nf(1/z)|\le a_n$ for $|z|\le 1$ by the Maximum Modulus Theorem. Hence, $|f(1/z)|\le a_n/|z|^n$ for $|z|\le 1$.

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Eneström-Kakeya Theorem Related Results

Theorem. (Joyal, Labell, Rahman 1967) If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|z| \leq (a_n - a_0 + |a_0|)/|a_n|$.

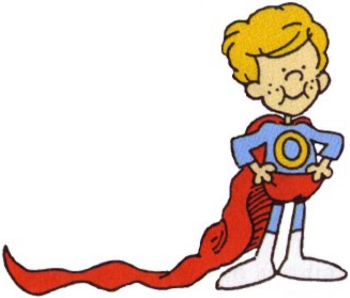
Theorem. (Gardner, Govil 1994) If $p(z) = \sum_{j=0}^{n} a_j z^j$, where $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$ for $j = 0, 1, 2, \ldots, n$. If

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \text{ and } \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n,$$

then all the zeros of p lie in $|z| \leq (|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n))/|a_n|$.

ETSU Students with Papers in This Area: Atif Abueida (2007), Jiencheng Cao (2003, 2004), Ty Frazier ("in the year 2525").

Number of Zeros Results



My Hero, Zero (January 13, 1973)

Titchmarsh's Number of Zeros Theorem

Jensen's Formula. (From Conway's *Function's of One Complex Variable I*, page 280.)
Let f be an analytic function on a region containing $\overline{B}(0;R)$ and suppose that $a_1, a_2, \ldots, a_n$ are the zeros of f in $B(0,R)$, repeated according to multiplicity. If $f(0) \neq 0$ then

$$\log|f(0)| = -\sum_{k=1}^{n} \log \frac{R}{|a_k|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta.$$

Titchmarsh's Number of Zeros Theorem. (Titchmarsh's *The Theory of Functions*, page 171.)
Let f be analytic in $|z| < R$. Let $|f(z)| \leq M$ in the disk $|z| \leq R$ and suppose $f(0) \neq 0$. Then for $0 < \delta < 1$ the number of zeros of $f(z)$ in the disk $|z| \leq \delta R$ is less than

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Titchmarsh's Theorem, Proof

Proof. Let f have n zeros in the disk $|z| \le \delta R$, say $a_1, a_2, \ldots, a_n$. Then for $1 \le k \le n$ we have $|a_k| \le \delta R$, or $\dfrac{R}{|a_k|} \ge \dfrac{1}{\delta}$. So

$$\sum_{k=1}^{n} \log \frac{R}{|a_k|} = \log \frac{R}{|a_1|} + \log \frac{R}{|a_2|} + \cdots + \log \frac{R}{|a_n|} \ge n \log \frac{1}{\delta}. \qquad (*)$$

By Jensen's Formula, we have

$$\begin{aligned} \sum_{k=1}^{n} \log \frac{R}{|a_k|} &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \, d\theta - \log |f(0)| \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \log M \, d\theta - \log |f(0)| \\ &= \log M - \log |f(0)| \\ &= \log \frac{M}{|f(0)|}. \qquad (**) \end{aligned}$$

Combining $(*)$ and $(**)$ gives...

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$$n \log \frac{1}{\delta} \leq \sum_{k=1}^{n} \log \frac{R}{|a_k|} \leq \log \frac{M}{|f(0)|},$$

or

$$n \leq \frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}.$$

Since n is the number of zeros of f in $|z| \leq \delta R$, the result follows. □

Number of Zeros, Students

Theorem

Pukhta 2011. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be such that* $|arg(a_j) - \beta| \leq \alpha \leq \pi/2$ *for all* $1 \leq j \leq n$ *and some real* α *and* β, *and*

$$0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{n-1}| \leq |a_n|.$$

Then the number of zeros of p *in* $|z| \leq \delta$, $0 \leq \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

ETSU Students with Papers in This Area: Brett Shields (2013, 2015), Derrick Bryant (2016).

Rate of Growth Results

Sergei Bernstein (1880–1968)

Application of the Maximum Modulus Theorem

Theorem

Rate of Growth, Bernstein. *If p is a polynomial of degree n such that $|p(z)| \leq M$ on $|z| = 1$, then for $R \geq 1$ we have*

$$\max_{|z|=R} |p(z)| \leq MR^n.$$

Proof. For $p(z) = \sum_{k=0}^{n} a_k z^k$ we have $r(z) = z^n p(1/z) = \sum_{k=0}^{n} a_k z^{n-k}$. Notice that for $|z| = 1$ (and $1/z = \overline{z}$) we have $\|r\| = \|p\|$ where $\|p\| = \max_{|z|=1} |p(z)|$. By the Maximum Modulus Theorem, for $|z| \leq 1$ we have $|r(z)| \leq \|r\| = \|p\| \leq M$. That is, $|z^n p(1/z)| \leq M$ for $|z| \leq 1$. Replacing z with $1/z$, we have $|(1/z^n)p(z)| \leq M$ for $|z| \geq 1$, or $|p(z)| \leq M|z|^n$ for $|z| \geq 1$. □

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Ankeny and Rivlin, Students

Theorem

Ankeny and Rivlin, 1955. *If p is a polynomial of degree n such that $p(z) \neq 0$ for $|z| < 1$ and $|p(z)| \leq M$ on $|z| = 1$, then for $R \geq 1$ we have*

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} M.$$

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Aziz and Dawood, 1988. *If p is a polynomial of degree n such that $p(z) \neq 0$ for $|z| < 1$ and $|p(z)| \leq M$ on $|z| = 1$, then for $|z| = R \geq 1$ we have*

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} M - \frac{R^n - 1}{2} \min_{|z|=1} |p(z)|.$$

ETSU Students with Papers in This Area: Amy Weems (2004, 2004).

Bernstein's Inequality

Sergei Bernstein (1880–1968)

Definition. For a polynomial p, define the *norm* $\|p\| = \max_{|z|=1} |p(z)|$. This is sometimes called the "sup norm" or "infinity norm" denoted $\|p\|_\infty$.

Note. Bernstein's Inequality in the complex setting states: "If p is a polynomial of degree n, then $\|p'\| \leq n\|p\|$. Equality holds if and only if $p(z) = \lambda z^n$ for some $\lambda \in \mathbb{C}$."

Note. Bernstein's original result (in 1926) concerned trigonometric polynomials, which are of the form $\sum_{v=-n}^{n} a_v e^{iv\theta}$. The version presented here is a special case of Bernstein's general result. There is a lengthy history of the so-called "Bernstein's Inequality" (there is also a different result in statistics with the same name).

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Lemma

Let p and q be polynomials such that (i) $\lim_{|z|\to\infty} |p(z)/q(z)| \le 1$*, (ii)* $|p(z)| \le |q(z)|$ *for* $|z| = 1$*, and (iii) all zeros of q lie in* $|z| \le 1$*. Then* $|p'(z)| \le |q'(z)|$ *for* $|z| = 1$*.*

Proof. Define $f(z) = p(z)/q(z)$. Then f is analytic on $|z| > 1$, $|f(z)| \le 1$ for $|z| = 1$, and $\lim_{|z|\to\infty} |f(z)| \le 1$. So by the Maximum Modulus Principle for Unbounded Domains,

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Note. This lemma as stated was proved by Bernstein. A generalization was proven where "$|z| < 1$" is replaced with a convex domain D and "$|z| = 1$" is replaced with the boundary of D, ∂D.

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Bernstein's Inequality

Theorem

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$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.$$

Proof. Let $M = \max_{|z|=1} |p(z)|$ and define $q(z) = Mz^n$. Then
(i) $|p(z)| \leq R^n M$ for $|z| = R$ by Bernstein's Rate of Growth Theorem, and so $\lim_{|z|=R\to\infty} |p(z)/q(z)| \leq \lim_{R\to\infty}(R^n M)/(R^n M) = 1$,
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□

Theorem

Erdös-Lax Theorem 1944. *Let p be a polynomial of degree n where* $p(z) \neq 0$ *for* $|z| < 1$. *Then* $\|p'(z)\|_\infty \leq \frac{n}{2}\|p(z)\|_\infty$.

Theorem

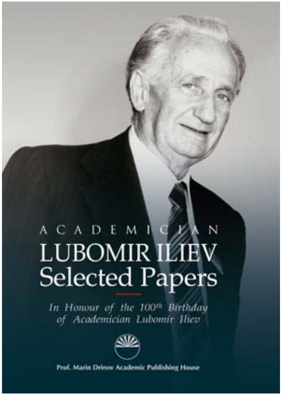
de Bruijn's Theorem 1947. *Let p be a polynomial of degree n where* $p(z) \neq 0$ *for* $|z| < 1$. *Then for* $1 \leq \delta \leq \infty$,

$$\|p'\|_\delta \leq \frac{n}{\|1+z\|_\delta}\|p\|_\delta,$$

where $\|p\|_\delta = \left\{\int_0^{2\pi} |p(e^{i\theta})|^\delta \, d\theta\right\}^{1/\delta}$ *for* $1 \leq \delta < \infty$.

ETSU Students with Papers in This Area: Amy Weems (1998).

Iliev-Sendov Conjecture



Ljubomir Iliev (1913–2000)

Blagovest Sendov (1932–)

Iliev-Sendov Conjecture

Note. The conjecture explored now is known variously as the Iliev Conjecture, the Iliev-Sendov Conjecture, and the Sendov Conjecture (making it particularly difficult to search for papers on the subject). It was originally posed by Bulgarian mathematician Blagovest Sendov in 1958 (according to some references; sometimes the year 1962 is given), but often attributed to Iliev because of a reference in Hayman's *Research Problems in Function Theory* in 1967. To muddle things further, sometimes "Iliev" is spelled "Ilieff."

Conjecture. The Iliev-Sendov Conjecture.
If all the zeros of a polynomial p lie in $|z| \leq 1$ *and if r is a zero of p, then there is a zero of* p' *in the circle* $|z - r| \leq 1$.

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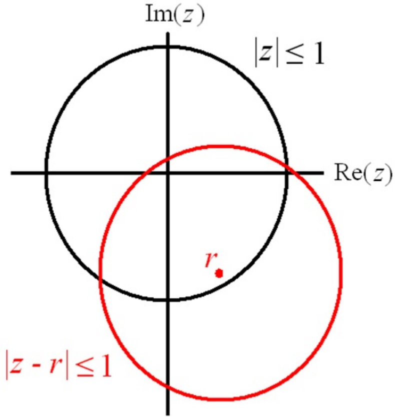
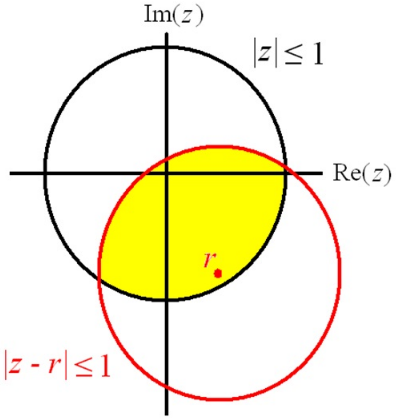
Iliev-Sendov Conjecture Picture

Note. Combining the Iliev-Sendov Conjecture with Corollary 3, we can further restrict the conjectured location of the critical points of p.





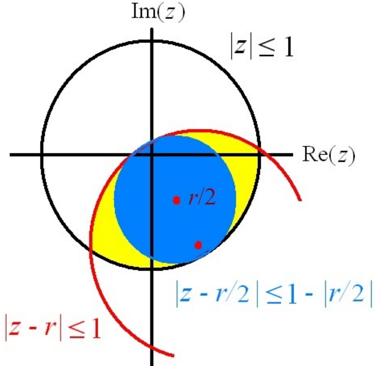
Iliev-Sendov Conjecture Special Cases

Note. According to a paper by Michael Miller (2008), there have been over 80 papers written on the conjecture. As a result, it has be demonstrated in many special cases. Some of the special cases are:

1. 3rd and 4th degree polynomials (Rubenstein 1968),
2. 5th degree polynomials (Meir and Sharma 1969),
3. polynomials having a root of modulus 1 (Rubenstein 1968, Schmeisser 1969),
4. polynomials with real and non-positive coefficients (Schmeisser 1971),
5. polynomials with at most three distinct zeros (Cohen and Smith 1988, Saff and Twomey 1971),
6. polynomials with at most six distinct zeros (Borcea 1996),
7. polynomials of degree less than or equal to 6 (Brown 1991),
8. polynomials of degree less than or equal to 8 (Brown 1999), and
9. the circle $|z - r| \leq 1.08331641$ (Bojanov, Rahman, Szynal 1985).

Goodman-Rahman-Ratti Conjecture

Note. A common approach to proving a difficult conjecture is to prove something even more restrictive than the conjecture, and then the conjecture falls as a corollary. In 1969, Goodman, Rahman, and Ratti (and independently Schmiesser in 1969) conjectured that the Iliev-Sendov Conjecture could be modified to the claim that (with the notation above) the region $|z - r/2| \leq 1 - |r|/2$ must contain a zero of p'. This is the blue region here:



Goodman-Rahman-Ratti Conjecture, continued

Note. However, this conjecture is not true as shown by Micheal Miller in 1990. The following eighth degree polynomial violates the Goodman, Rahman, Ratti Conjecture:

$$p(z) = (z - 0.8)(z^7 + 1.241776468z^6 + 1.504033112z^5 + 1.702664563z^4$$
$$+1.702664563z^3 + 1.504033112z^2 + 1.241776468z + 1).$$

Miller also found degree 6, 10, and 12 polynomials violating the new conjecture.

Complex Analysis 1 in Fall 2017!!!

The ETSU Department of Mathematics and Statistics will offer Complex Analysis 1 (MATH 5510) in fall semester (TR 2:15–3:35). Topics to be covered include analytic functions, Möbius transformations, power series, zeros, complex integration, and the many versions of Cauchy's Theorem. Additional topics to be covered in Complex Analysis 2 will be the Open Mapping Theorem, singularities, Laurent Series, the Argument Principle, the Maximum Modulus Theorem, and Schwarz's Theorem. Time permitting, spaces of functions and analytic continuation will be covered.

The syllabus for the class online is at:
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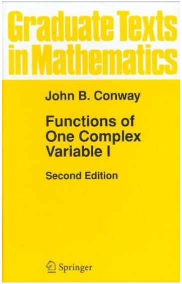

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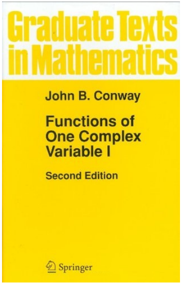

References 1

General References

1. L. Ahlfors, *Complex Analysis*, International Series in Pure & Applied Mathematics (Book 7), McGraw-Hill (1979).
2. J. Conway, *Functions of One Complex Variable I*, 2nd Edition, Graduate Texts in Mathematics, Vol. 11, Springer-Verlag (1978).
3. M. Marden, *Geometry of Polynomials*, Revised Edition, Mathematical Surveys and Monographs Book 3, American Mathematical Society (1970).
4. E. Titchmarsh, *The Theory of Functions*, 2nd Editon, Oxford University Press (1939), corrected printing 1968.

References 2

Eneström-Kakeya Theorem References

1. G. Eneström, Härledning af an allmän formel för antalet pensionärer, som vid en godtyeklig tidpuckt förefinnas inom en sluten pensionsclassa. Öfvers, Vetensk.-Akad Förh., **50**, 405–415 (1893).
2. R. Gardner and N. K. Govil, On the Location of the Zeros of a Polynomial, *Journal of Approximation Theory*, **78** (1994) 286–292.
3. A. Joyal, G. Labelle, Q. I. Rahman, On the Location of Zeros fo Polynomials, *Canadian Math. Bulletin*, **10**(1), 53–63 (1967).
4. S. Kakeya, On the Limits of the Roots of an Algebraic Equation with Positive Coefficients, *Tôhoku Math. J. First Ser.*, **2**, 140–142 (1912-1913).

References 3

Rate of Growth References

1. N. C. Ankeny and T. J. Rivlin, On a Theorem of S. Bernstein, *Pacific Journal of Mathematics*, **5** (1955), 849–852.
2. A. Aziz and M. Dawood, Inequalities for a Polynomial and its Derivative, *Journal of Approximation Theory*, **53** (1988), 155–162.

Number of Zeros References

1. M. S. Pukhta, On the Zeros of a Polynomial, *Applied Mathematics*, **2** (2011), 1356–1358.

References 4

Bernstein Inequality References

1. S. Bernstein, *Sur L'prdre de la meilleur approximation des fonctions continues par des polynômes de degré donné*, Memoire d el'Académie Royal de Belgique 1912, **4**(2): 1–103.
2. N. de Bruijn, Inequalities Concerning Polynomials in the Complex Domain, *Nederl. Akad. Wetensch. Proc.* **50** (1947), 1265–1272; *Indag. Math.* **9** (1947), 591–598.
3. P. D. Lax, Proof of a Conjecture of P. Erdös on the Derivative of a Polynomial, *Bulletin of the American Matheamtical Society*, **50** (1944), 509–513.

References 5

Illiev-Sendov Conjecture References

1. B. Bojanov, Q. Rahman, and J. Szynal, "On a Conjecture of Sendov about the Critical Points of a Polynomial," *Mathematische Zeitschrift*, **190**, 281–285 (1985).
2. I. Borcea, "On the Sendov Conjecture for Polynomials with at Most Six Distinct Roots," *Journal of Mathematical Analysis and Applications*, **200**, 182–206 (1996).
3. J. Brown, "On the Sendov Conjecture for Sixth Degree Polynomials," *Proceedings of the American mathematical Society*, **113**(4), 939–946 (1991).
4. J. Brown, and G. Xiang, "Proof of the Sendov Conjecture for Polynomials of Degree at Most Eight, *Journal of Mathematical Analysis and Applications*, **232**, 272–292 (1999).
5. G. Cohen and G. Smith, "A Simple Verification of Ilieff's Conjecture for Polynomials With Three Zeros," *The American Mathematical Monthly*, **95**(8), 734–737 (1988).
6. A. Goodman, Q. I. Rahman, and J. Ratti, "On the Zeros of a Polynomial and its Derivative," *Proc. A.M.S.*, **21**, 273–274 (1969).

References 6

7. W. Hayman, *Research Problems in Function Theory*, London: Athlone (1967).
8. A. Meir and A. Sharma, "On Ilyeff's Conjecture," *Pacific Journal of Mathematics*, **31**, 459–467, (1969).
9. M. Miller, "Maximal Polynomials and the Illieff-Sendov Conjecture, *Transactions of the American Mathematical Society*, **321**(1), 285–303 (1990).
10. M. Miller, "Unexpected Local Extrema for the Sendov Conjecture," *Journal of Mathematical Analysis and Applications*, **348**, 461–468 (2008).
11. Z. Rubenstein, "On a Problem of Ilyeff," *Pacific Journal of Mathematics*, **26**, 159–161 (1968).
12. E. Saff and J. Twomey, "A Note on the Location of Critical Points of Polynomials," *Proc. A.M.S.*, **27**, 303-308 (1971)
13. G. Schmeisser, "Bemerkungen zu einer Vermutung von Ilieff," *Mathematische Zeitschrift*, **111**, 121–125 (1969).
14. G. Schmeisser, "Zur Lage der Kritichen punkte eines polynoms," *Rend. Sem. Mat. Univ. Padova*, **46**, 405–415 (1971).

References 7

Publications by ETSU Students

1. Atif Abueida and R. Gardner, Some Generalizations of the Enestrom-Kakeya Theorem, *Theory of Polynomials and Applications, in Theory of Polynomials and Applications, Recent Advances*, ed. K.K. Dewan, New Dehli: Deep & Deep Publications (2007), 1-10.
2. Derrick Bryant and R. Gardner, Results on the Number of Zeros in a Disk for Three Types of Polynomials, *Acta et Commentationes Universitatis Tartuensis de Mathematica*, **20**(2) (2016), 135-149.
3. Jiancheng Cao and R. Gardner, Restrictions on the Zeros of a Polynomial as a Consequence of Conditions on the Coefficients of Even and Odd Powers of the Variable, *Journal of Computational and Applied Mathematics*, **155**(1) (2003) 153-162.
4. R. Gardner and Amy Weems, A Bernstein Type L^p Inequality for a Certain Class of Polynomials, *Journal of Mathematical Analysis and its Applications*, **219** (1998) 472-478.
5. R. Gardner, N. K. Govil, and Amy Weems, Growth of Polynomials not Vanishing Inside a Circle, *International Journal of Pure and Applied Mathematics*, **13**(4) (2004), 491-498.

References 8

Publications by ETSU Students, Continued

6. R. Gardner, N. K. Govil, and Amy Weems, Some Results Concerning Rate of Growth of Polynomials, *East Journal of Approximations*, 10(3) (2004), 301-312.
7. R. Gardner and Brett Shields, The Number of Zeros of a Polynomial in a Disk, *Journal of Classical Analysis*, **3**(2) (2013), 167-176.
8. R. Gardner and Brett Shields, The Number of Zeros of a Polynomial in a Disk as a Consequence of Restrictions on the Coefficients, *Acta et Commentationes Universitatis Tartuensis de Mathematica*, **19**(2) (2015), 109-120.
9. A. Lin, B. Huang, Jiancheng Cao, R. Gardner, Some Results on the Location of Zeros of Analytic Functions, *Chinese Journal of Engineering Mathematics*, **21**(5) (2004), 785-791.

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