BERNSTEIN INEQUALITIES FOR POLYNOMIALS AND ENTIRE FUNCTIONS

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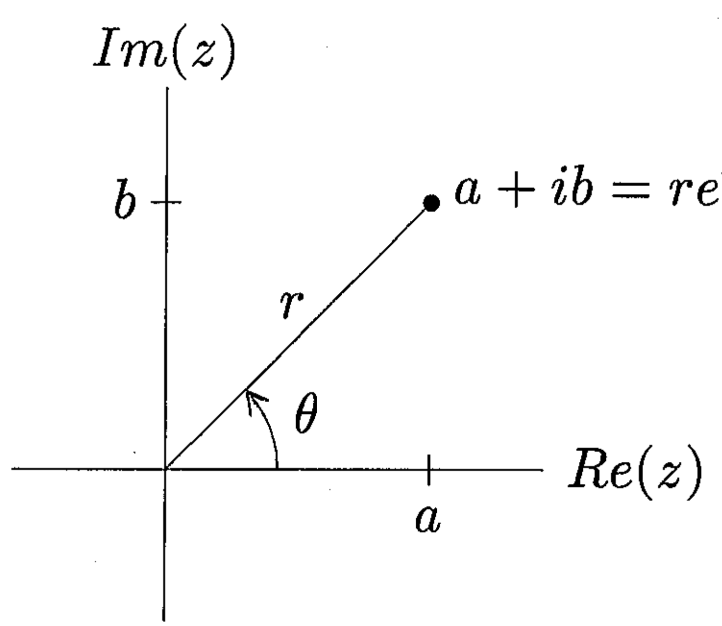
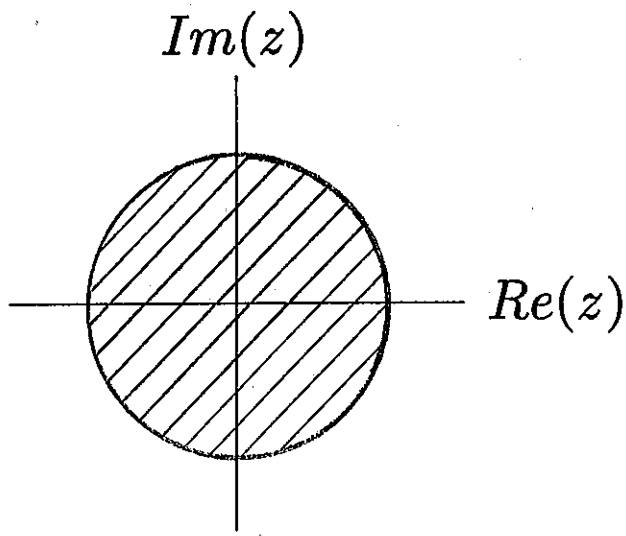
INTRODUCTION

Definition. A *complex number* is one of the form $z = a + ib$ where a and b are real and $i^2 = -1$. a is called the *real part* of z, denoted $Re(z)$, and b is called the *imaginary part* of z, denoted $Im(z)$. The *modulus* of $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$.

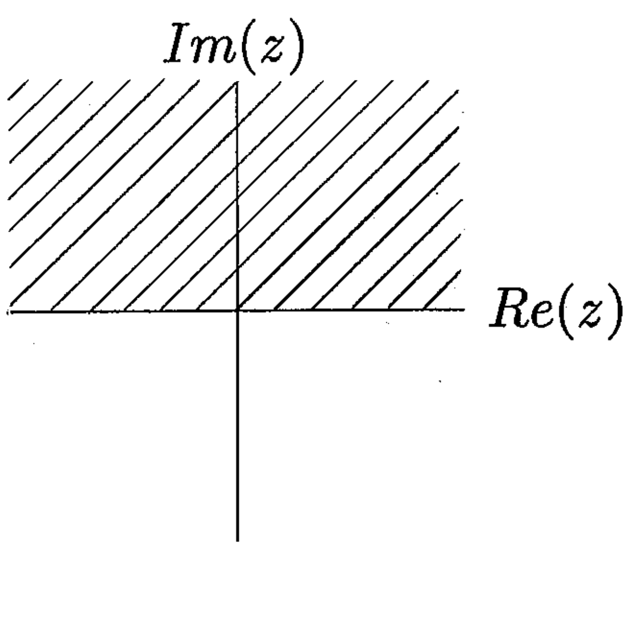
Note. We can plot $z = a + ib$ in the *complex plane*:



Note. The graph of $|z| \leq 1$ is:



Note. The graph of $Im(z) \geq 0$ is:



HISTORY

Note. Mendeleev studied the specific gravity of a solution as a function of the percentage of the dissolved substance. After plotting data, he asked the question:
If $p(x)$ is a real polynomial function of degree two and $|p(x)| \leq 1$ on $[-1,1]$, how large can $|p'(x)|$ be on $[-1,1]$?
Answer: $|p'(x)| \leq 4$. The result is "sharp" (or best possible) as is shown by the example $p(x) = 1 - 2x^2$.

In general:
Theorem (Markov). If $p(x)$ is a real polynomial of degree n, and $|p(x)| \leq M$ on $[-1,1]$ then $|p'(x)| \leq Mn^2$ on $[-1,1]$.

Note. This result is sharp for the n^{th} Chebyshev polynomial, $p(x) = \pm T_n(x) = \pm \cos(n \cos^{-1}(x))$.

RESULTS FOR COMPLEX POLYNOMIALS

Definition. Let $\mathcal{P}_n$ denote the class of all polynomials of degree less than or equal to n. For $P \in \mathcal{P}_n$, define the *norm* $\|P\| = \max_{|z|=1} |P(z)|$.

Theorem (Bernstein). If $P \in \mathcal{P}_n$, then

$$\|P'\| \le n\|P\|.$$

Note. This result is sharp if and only if $P(z) = \alpha z^n$.

Theorem (Erdös-Lax). If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$ then

$$\|P'\| \leq \frac{n}{2}\|P\|.$$

Note. This result is sharp for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

Theorem (Malik). If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < K$, $K \geq 1$, then

$$\|P'\| \leq \frac{n}{1+K}\|P\|.$$

Note. This result is also sharp, as is shown by the example $P(z) = \left(\frac{z+K}{1+K}\right)^n$.

Note. With $K = 1$, Malik's theorem reduces to the Erdös-Lax theorem.

Theorem (Govil and Labelle). If

$$P(z) = a_n \Pi_{v=1}^{n}(z - z_v) \in \mathcal{P}_n$$

and $|z_v| \geq K_v \geq 1$ then

$$\|P'\| \leq \frac{n}{2}\left(1 - \frac{1}{1 + \frac{2}{n}\Sigma_{v=1}^{n}\frac{1}{K_v - 1}}\right)\|P\|.$$

Note. This result reduces to Malik's theorem if $K_v \geq K \geq 1$ for all v, and reduces to the Erdös-Lax result if $K_v = 1$ for some v.

Definition. For $P \in \mathcal{P}_n$, define the L^p *norm* of P as

$$\|P\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}.$$

Theorem (DeBruijn). If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$ then for $p \geq 1$,

$$\|P'\|_p \leq \frac{n}{\|1 + z^n\|_p} \|P\|_p.$$

Note. If we let $p \to \infty$, DeBruijn's theorem reduces to the Erdös-Lax result.

Theorem (Gardner and Govil). Suppose $P(z) = a_n \Pi_{v=1}^{n} (z - z_v) \in \mathcal{P}_n$ and $|z_v| \geq K_v \geq 1$. Then for $p \geq 1$,

$$\|P'\|_p \leq \frac{n}{\|t_0 + z^n\|_p} \|P\|_p$$

where

$$t_0 = \left(\frac{\sum_{v=1}^{n} \frac{K_v}{K_v - 1}}{\sum_{v=1}^{n} \frac{1}{K_v - 1}} \right) = 1 + \frac{n}{\sum_{v=1}^{n} \frac{1}{K_v - 1}}.$$

Note. If $K_v = 1$ for any v, this result reduces to DeBruijn's theorem. If $p \to \infty$, this result reduces to the Govil-Labelle theorem.

Note. Each of the above results involving L_p norms can be extended to the case $p \in [0, 1)$ where

$$\|P\|_0 = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| \, d\theta\right).$$

Definition. For $P \in \mathcal{P}_n$ define *polar derivative of P with respect to the pole ζ* as

$$D_\zeta[P(z)] = nP(z) + (\zeta - z)P'(z).$$

Theorem (Laguerre). Let $P \in \mathcal{P}_n$ and suppose that all the zeros of P lie in a generalized circular region C. If z is any zero of $D_\zeta[P]$ then not both points z and ζ lie outside of C.

Theorem (Aziz). If $P \in \mathcal{P}_n$, and $P(z) \neq 0$ in $|z| < K$ where $K \geq 1$ then for $|\zeta| \geq 1$

$$\|D_\zeta[P]\| \leq n\left(\frac{K+|\zeta|}{1+K}\right)\|P\|.$$

Note. If $\zeta \to \infty$ then this result implies Malik's theorem.

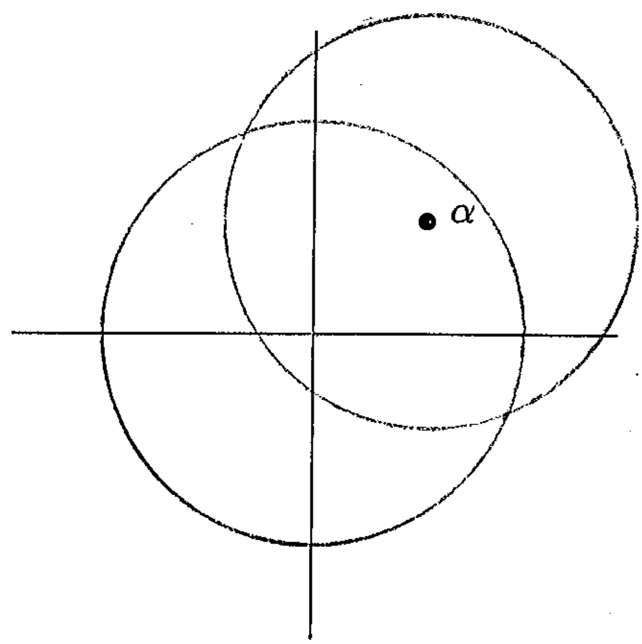
Corollary (to Laguerre's Theorem). Suppose all the zeros of the polynomial P lie in $|z| \leq 1$. Then all the zeros of P' lie in $|z| \leq 1$.

proof. Choose ζ such that $|\zeta| > 1$. Then all the zeros of $D_\zeta[P]$ and $\dfrac{D_\zeta[P]}{\zeta}$ lie in $|z| \leq 1$. Therefore all the zeros of

$$\lim_{\zeta\to\infty}\left(\frac{D_\zeta[P]}{\zeta}\right) = \lim_{\zeta\to\infty}\frac{nP(z)+(\zeta-z)P'(z)}{\zeta} = P'(z)$$

lie in $|z| \leq 1$. ∎

Conjecture (Illief-Sendov). Suppose all the zeros of the polynomial $P(z)$ lie in $|z| \leq 1$ and let α be one such zero. Then there is a zero, say β, of $P'(z)$ such that $|\alpha - \beta| \leq 1$.



ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

Definition. Let $\mathbf{C}$ represent the complex plane and suppose $F : \mathbf{C} \to \mathbf{C}$. Then F is *analytic at a point* if it is differentiable in an open neighborhood of that point. A function analytic at all points of $\mathbf{C}$ is an *entire function.*

Note. Some facts about entire functions:

1. If $F(z)$ has a finite number of zeros then $F(z) = P(z)g(z)$ where $P(z)$ is a polynomial, $g(z)$ is entire and $g(z)$ is never 0.

2. **(Weierstrass)** Every entire function F (not identically 0) can be represented as

$$F(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{P_n(z)}$$

where the product is taken over the zeros of $F(z)$ other than $z = 0$, $g(z)$ is entire and P_n is a polynomial.

Definition. An entire function F is said to be *of exponential type* if the inequality $|F(z)| \le Ae^{B|z|}$ holds for some positive A and B and for all z.

Definition. Let $M(F,r) = \max_{|z|=r} |F(z)|$. The *order* ρ of F is

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(F,r)}{\log r}.$$

An entire function of positive order ρ is of *type* τ if

$$\tau = \limsup_{r\to\infty} \frac{\log M(F,r)}{r^\rho}.$$

Definition. An entire function is said to be of *exponential type* τ if either it is of order 1 and type less than or equal to τ, or it is of order less than 1. Denote the class of all such functions by $\mathcal{E}_\tau$.

Definition. For $F \in \mathcal{E}_\tau$, let the *norm* of F be $\|F\| = \sup_{x \in R} |F(x)|$. Define the *indicator function* of F as

$$h_F(\theta) = \limsup_{r \to \infty} \frac{\log |F(re^{i\theta})|}{r}.$$

Example. The function $F(z) = e^{\tau z^\rho}$ is of order ρ and type τ.

Example. Consider $F(z) = \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$. Then

$$\frac{e^R - 1}{2} \leq \max_{|z|=R} |\sin(z)| \leq \frac{e^R + 1}{2}.$$

So $F(z) = \sin(z)$ is of order $\rho = 1$ and type $\tau = 1$. Also $h_F\left(\frac{\pi}{2}\right) = 1$.

Note. Some facts about $\mathcal{E}_T$:

1. The Phragmen-Lindelöf Theorem is, in a sense, a generalization of the Maximum Modulus Theorem. From it, we have:

Theorem. If $F \in \mathcal{E}_T$, $h_F\left(\frac{\pi}{2}\right) \leq c$ and $\|F\| \leq M$ then for every z with $y = Im(z) \geq 0$,

$$|F(z)| \leq Me^{cy}.$$

2. $\mathcal{E}_T$ is a Banach Space with our norm $\|\cdot\|$.

3. Consider

$$\mathbf{P} = \left\{F \in \mathcal{E}_\pi : \int_{-\infty}^{\infty} |F(x)|^2 \, dx < \infty\right\}.$$

This is called the *Paley-Wiener Space.* If $F \in \mathbf{P}$ then there exists $\Phi \in L^2[-\pi, \pi]$ such that

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(t) e^{izt} \, dt = \mathcal{F}^{-1}\{\Phi\} \text{ and}$$

$$\Phi(t) = \int_{-\infty}^{\infty} F(x) e^{-ixt} \, dx = \mathcal{F}\{F\}.$$

Since the Fourier transform $\mathcal{F}$ is an isometry (preserves norms), **P** is isometrically isomorphic to $L^2[-\pi, \pi]$. Also, **P** is a Hilbert Space with the inner product $(f, g) = \int_{-\infty}^{\infty} f\overline{g}\, dx$. If $f \in \mathbf{P}$ then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \frac{\sin \pi(z-n)}{\pi(z-n)} \qquad (*)$$

with $\sum |c_n|^2 < \infty$. That is,

$$\left\{ \frac{\sin \pi(z-n)}{\pi(z-n)} \right\}_{n \in Z}$$

is an (orthonormal) Riesz basis for **P**. Setting $z = n$ in $(*)$, we have $c_n = f(n)$. So

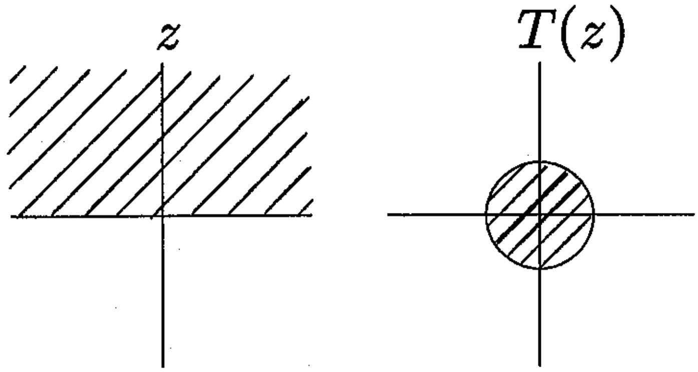
$$f(z) = \sin \pi z \sum_{n=-\infty}^{\infty} (-1)^n \frac{f(n)}{\pi(z-n)}.$$

So if $f \in \mathbf{P}$ and f is known on the integers, then f is determined. This is called the Shannon Sampling Theorem.

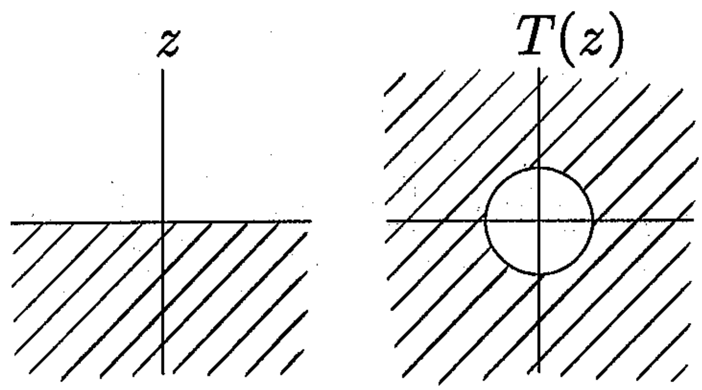
Note. Consider the transformation $T : z \to e^{iz}$. This transformation satisfies the following properties:

1. $T : \mathbf{R} \to \{z \mid |z| = 1\}$

2. $T : \{z \mid Im(z) > 0\} \to \{z \mid |z| < 1\}$



3. $T : \{z \mid Im(z) < 0\} \to \{z \mid |z| > 1\}$



Note. We want to consider Bernstein type results for the class $\mathcal{E}_\tau$.

Theorem (Bernstein). If $F \in \mathcal{E}_\tau$ then

$$\|F'\| \leq \tau \|F\|.$$

Note. Bernstein's theorem for polynomials can be derived from this result by considering $P(T(z)) = P(e^{iz}) \in \mathcal{E}_n$ where $P \in \mathcal{P}_n$.

Theorem (Boas). If $F \in \mathcal{E}_\tau$, $h_F\left(\frac{\pi}{2}\right) = 0$ and $f(z) \neq 0$ for $Im(z) > 0$ then

$$\|F'\| \leq \frac{\tau}{2}\|F\|.$$

Note. The Erdös-Lax theorem can be derived from this.

Definition. For $F \in \mathcal{E}_\tau$ define the *polar derivative of F with respect to ζ* as

$$D_\zeta[F(z)] = \tau F(z) + i(1-\zeta)F'(z).$$

Theorem (Rahman-Schmeisser). Let $F \in \mathcal{E}_\tau$ where $\tau > 0$ and $h_F(\frac{\pi}{2}) = 0$. Let H denote the (closed or open) upper half plane. If $F(z) \neq 0$ for $z \in H$ then $D_\zeta[F] \neq 0$ for $z \in H$ and $|\zeta| \leq 1$.

Note. Laguerre's theorem can be derived from this result.

Corollary. Let $F \in \mathcal{E}_\tau$ where $\tau > 0$ and $h_F(\frac{-\pi}{2}) = \tau$. Let L denote the (closed or open) lower half plane. If $F(z) \neq 0$ for $z \in L$ then $D_\zeta[F] \neq 0$ for $z \in L$ and $|\zeta| \geq 1$.

Note. If $\zeta \to \infty$ this means $F'(z) \neq 0$ for $z \in L$.

Theorem (Gardner and Govil). Let $F \in \mathcal{E}_\tau$, $h_F(\frac{\pi}{2}) = 0$ and $F(z) \neq 0$ in H. Then for $z \in L$ and $|\zeta| \geq 1$,

$$|D_\zeta[F(z)]| \leq \frac{\tau}{2}\left(|\zeta|e^{\tau|y|} + 1\right)\|F\|$$

where y=$Im(z)$.

Note. This result implies the Boas result for entire functions when $\zeta \to \infty$.

Theorem (Gardner and Govil). Let $F \in \mathcal{E}_\tau$, $h_F(\frac{\pi}{2}) = 0$ and suppose $F(z)$ has all its zeros in $Im(z) \leq k \leq 0$. If $|\zeta| \geq 1$, $h_{D_\zeta[F]}\left(\frac{\pi}{2}\right) \leq -c < 0$ and $h_{D_\zeta[G]}\left(\frac{\pi}{2}\right) \leq -c < 0$ where $G(z) = e^{i\tau z}\overline{F(\overline{z})}$ then

$$\|D_\zeta[F]\| \leq \frac{\tau(|\zeta| + 1)}{e^{c|k|} + 1}\|F\|.$$

Note. Letting $\zeta \to \infty$, this theorem reduces to a result of Govil and Rahman.