ON THE LOCATION OF THE ZEROS OF A POLYNOMIAL

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HISTORY

Question. If $p(z)$ is a polynomial of degree n, then what are the zeros of $p(z)$?

Answer 1. If $n = 1, 2, 3,$ or 4, then no problem (antiquity).

Answer 2. If $n \geq 5$, then **problem!** (Abel *et al.*)

Question. If $p(z) = \sum_{v=0}^{n} a_v z^v$, then what restrictions can be put on the location of the zeros of $p(z)$ in the complex plane?

Theorem 1. Cauchy 1830
All the zeros of $p(z) = \sum_{v=0}^{n} a_v z^v$, where $a_n \neq 0$, lie in the circle $|z| < 1+M$, where $M = \max_{0\le j\le(n-1)} \left|\frac{a_j}{a_n}\right|$.

Theorem 2. Kakeya-Hayashi-Hurwitz 1910
All the zeros of $p(z) = \sum_{v=0}^{n} a_v z^v$, where a_j are real and positive for $j = 0, 1, \ldots, n$, lie in

$$R_1 \le |z| \le R_2$$

where $R_1 = \min_{0\le j\le(n-1)} \left(\frac{a_j}{a_{j+1}}\right)$ and $R_2 = \max_{0\le j\le(n-1)} \left(\frac{a_j}{a_{j+1}}\right)$.

Theorem 3. Eneström-Kakeya 1920
If $p(z) = \sum\limits_{v=0}^{n} a_v z^v$ is a polynomial of degree n with real coefficients satisfying

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_n,$$

then all the zeros of $p(z)$ lie in $|z| \leq 1$.

Proof. Consider the polynomial

$$P(z) = (1-z)p(z) = a_0 + \sum_{j=1}^{n}(a_j - a_{j-1})z^j - a_n z^{n+1} \equiv -a_n z^{n+1} + G_2(z).$$

If $|z| > 1$ then

$$\begin{aligned}
|G_2(z)| &= |a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n| \\
&\leq |a_0||z|^n + |a_1 - a_0||z|^n + |a_2 - a_1||z|^n + \cdots + |a_n - a_{n-1}||z|^n \\
&= |z|^n(|a_0| + |a_1 - a_0| + |a_2 - a_1| + \cdots + |a_n - a_{n-1}|) \\
&= |z|^n(a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1})) \\
&= a_n|z|^n.
\end{aligned}$$

So for $|z| > 1$

$$\begin{aligned}
|P(z)| &\geq |-a_n||z|^{n+1} - |G_2(z)| \\
&\geq a_n|z|^{n+1} - a_n|z|^n = a_n|z|^n(|z| - 1) > 0.
\end{aligned}$$

Therefore for $|z| > 1$, $P(z) \neq 0$ and in turn $p(z) \neq 0$. ∎

Theorem 4. Joyal-Labelle-Rahman 1967
If $p(z) = \sum\limits_{v=0}^{n} a_v z^v$ is a polynomial of degree n with real coefficients, $a_n \neq 0$, satisfying

$$a_0 \leq a_1 \leq \cdots \leq a_n,$$

then all the zeros of $p(z)$ lie in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Note. If $a_0 > 0$, then Theorem 4 reduces to Theorem 3.

Proof. Consider the polynomial

$$P(z) = (1-z)p(z) = a_0 + \sum_{j=1}^{n}(a_j - a_{j-1})z^j - a_n z^{n-1} \equiv -a_n z^{n+1} + G_2(z).$$

Then for $|z| = 1$,

$$\begin{aligned} |G_2(z)| &= |a_0 + \sum_{j=1}^{n}(a_j - a_{j-1})z^j| \\ &\leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_1 - a_0| + |a_0| \\ &= (a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_1 - a_0) + |a_0| \\ &= a_n - a_0 + |a_0|. \end{aligned}$$

Therefore

$$\left| z^n G_2\left(\frac{1}{z}\right) \right| \leq a_n - a_0 + |a_0| \text{ for } |z| = 1.$$

Since $z^n G_2\left(\frac{1}{z}\right)$ is analytic in $|z| \leq 1$, it follows that by the Maximum Modulus Theorem that

$$\left| G_2\left(\frac{1}{z}\right) \right| \leq \frac{a_n - a_0 + |a_0|}{|z|^n} \text{ for } |z| \leq 1.$$

Replacing z by $\frac{1}{z}$ we get

$$|G_2(z)| \le (a_n - a_0 + |a_0|)|z|^n \text{ for } |z| \ge 1.$$

Hence, if $|z| > \frac{(a_n - a_0 + |a_0|)}{|a_n|}$ then

$$\begin{aligned} |P(z)| &= |-a_n z^{n+1} + G_2(z)| \\ &\ge |-a_n||z|^{n+1} - |G_2(z)| \\ &\ge |a_n||z|^{n+1} - (a_n - a_0 + |a_0|)|z|^n \\ &= |z|^n(|a_n||z| - (a_n - a_0 + |a_0|)) > 0. \end{aligned}$$

Therefore for $|z| > \frac{a_n - a_0 + |a_0|}{|a_n|}$, $P(z) \neq 0$ and in turn $p(z) \neq 0$. ∎

Note. The following result for *analytic functions* has a rather flexible condition on the coefficients of the series expansion of the function.

Theorem 5. Aziz-Mohammed 1980
Let $f(z) = \sum_{v=0}^{\infty} a_v z^v$ be analytic in $|z| \leq t$. If $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$ for $j = 0, 1, \ldots$, for some k and r, and for some $t \geq 0$,

$$0 < \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots$$

and

$$\beta_0 \leq t\beta_1 \leq \cdots \leq t^r \beta_r \geq t^{r+1}\beta_{r+1} \geq \cdots$$

then $f(z) \neq 0$ in

$$|z| < \frac{t|a_0|}{2(\alpha_k t^k + \beta_r t^r) - (\alpha_0 + \beta_0)}.$$

Note. We put a similar conditions on the coefficients of a *polynomial.*

NEW RESULTS

Theorem A. Gardner and Govil 1994 *JAT*

Suppose $p(z) = \sum_{v=0}^{n} a_v z^v$, $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$ for $j = 0, 1, \ldots, n$, $a_n \neq 0$ and for some k and r and for some $t \geq 0$, we have

$$\alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq t^{k+2}\alpha_{k+2} \geq \cdots \geq t^n\alpha_n$$

and

$$\beta_0 \leq t\beta_1 \leq t^2\beta_2 \leq \cdots \leq t^r\beta_r \geq t^{r+1}\beta_{r+1} \geq t^{r+2}\beta_{r+2} \geq \cdots \geq t^n\beta_n.$$

Then $p(z)$ has all its zeros in $R_1 \leq |z| \leq R_2$ where

$$R_1 = \min\left\{t|a_0| \Big/ \Big(2(t^k\alpha_k + t^r\beta_r) - (\alpha_0 + \beta_0) - t^n(\alpha_n + \beta_n - |a_n|)\Big)\ , t\right\}$$

and

$$\begin{aligned} R_2 \;=\; \max\Bigg\{\Bigg(&|a_0|t^{n+1} - t^{n-1}(\alpha_0 + \beta_0) - t(\alpha_n + \beta_n) \\ &+(t^2+1)(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r) \\ &+(t^2-1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right) \\ &+(1-t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right)\Bigg) \Big/ |a_n|\, , \frac{1}{t}\Bigg\}. \end{aligned}$$

Proof of inner radius. Consider

$$P(z) = (t-z)p(z) = ta_0 + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \equiv ta_0 + G_1(z).$$

On $|z| = t$,

$$\begin{aligned}
|G_1(z)| &\leq \sum_{j=1}^{k}(t\alpha_j - \alpha_{j-1})t^j + \sum_{j=k+1}^{n}(\alpha_{j-1} - t\alpha_j)t^j \\
&\quad + \sum_{j=1}^{r}(t\beta_j - \beta_{j-1})t^j + \sum_{j=r+1}^{n}(\beta_{j-1} - t\beta_j)t^j + |a_n|t^{n+1} \\
&= -t(\alpha_0 + \beta_0) + 2(t^{k+1}\alpha_k + t^{r+1}\beta_r) - t^{n+1}(\alpha_n + \beta_n - |a_n|) \\
&\equiv M_1.
\end{aligned}$$

Applying Schwarz's Lemma to $G_1(z)$, we get

$$|G_1(z)| \leq \frac{M_1|z|}{t} \text{ for } |z| \leq t.$$

Which implies

$$|P(z)| = |-ta_0 + G_1(z)| \geq t|a_0| - |G_1(z)| \geq t|a_0| - \frac{M_1|z|}{t}.$$

Hence if $|z| < \min\left\{\frac{t^2|a_0|}{M_1}, t\right\} \equiv R_1$ then $P(z) \neq 0$ and in turn $p(z) \neq 0$.

∎

EXAMPLE. Suppose

$$\begin{aligned} p(z) &= (1+i) + (0.2+0.2i)z + (0.03+0.03i)z^2 \\ &\quad +(0.0031+0.0031i)z^3 + (0.0003+0.0003i)z^4. \end{aligned}$$

Then by Theorem 1 (of Cauchy), all the zeros of $p(z)$ lie in $|z| < 3334.33$. According to Theorem 5 (of Aziz and Mohammed, with $k = r = 0$ and $t = 5$), no zero of $p(z)$ lies in $|z| < 3.54$. Applying Theorem A, we find that the zeros of $p(z)$ lie in

$$3.74 \leq |z| \leq 23.15.$$

The inner radius is based on $k = r = 0$ and $t = 5$. The outer radius is based on $k = r = 0$ and $t = .0467$.

Then according to Theorem 1.3, $p(z)$ has no zeros in $|z| < 1.3598$. Applying Theorem 2.1 to $p(z)$ with $t = 10$, $k = r = 3$, we find that $p(z) \neq 0$ in $|z| < 1.6363$, an improvement over the bound from Theorem 1.3 of over 20%.

By making certain choices of t, k and r in Theorem A we obtain the following corollaries. In each, $p(z) = \sum_{v=0}^{n} a_v z^v$, $a_n \neq 0$, $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$ for $j = 0, 1, \ldots, n$.

If in Theorem A, we take $t = 1$, $k = n$ and $r = n$, then we get:

Corollary 1. If $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$ then $p(z)$ has all its zeros in

$$\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.$$

Notice that with $\beta_i = 0$ for $i = 0, 1, \ldots, n$, Corollary A implies the Eneström-Kakeya Theorem and also gives a zero free region about the origin. Namely, under these conditions, Corollary 1 restricts the zeros of $p(z)$ to

$$\frac{|a_0|}{|a_n| + a_n - a_0} \leq |z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

If in Theorem A, we take $t = 1$, $k = 0$ and $r = 0$ then we get:

Corollary 2. If $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ then $p(z)$ has all its zeros in

$$\frac{|a_0|}{|a_n| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)}{|a_n|}.$$

If in Theorem A, we take $t = 1$, $k = n$ and $r = 0$ then we get:

Corollary 3. If $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$ then $p(z)$ has all its zeros in

$$\frac{|a_0|}{|a_n| + \alpha_0 - \beta_0 - \alpha_n + \beta_n} \leq |z| \leq \frac{|a_0| + \alpha_0 - \beta_0 - \alpha_n + \beta_n}{|a_n|}.$$

Lastly, if in Theorem A, we take $t = 1$, $k = 0$ and $r = n$ then we get:

Corollary 4. If $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ then $p(z)$ has all its zeros in

$$\frac{|a_0|}{|a_n| - \alpha_0 + \beta_0 + \alpha_n - \beta_n} \leq |z| \leq \frac{|a_0| - \alpha_0 + \beta_0 + \alpha_n - \beta_n}{|a_n|}.$$