A MATHEMATICIAN LOOKS AT THE PHASE PLANE

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A Brief Survey of ODEs

Definition. An *ordinary differential equation* (ODE) is a relation among an independent variable x, an (unkown) function $y(x)$ of that variable, and certain of its derivatives. More precisely, an ODE is a function of $n+2$ variables evaluated at the independent variable t, the function y, and the n derivatives of y, set equal to a constant:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

A function y satisfying this equation is a ***solution***. The highest order of derivative of y appearing in the equation is the ***order*** of the ODE.

eg. *Bessel's Equation* is $x^2y'' + xy' + (x^2 - p^2)y = 0$. This arises in mechanics and electromagnetic theory. This is an ODE of order 2. This DE is used to define the ***Bessel Functions*** which have been called "the most important functions beyond the elementary ones." Solving the Bessel equation leads to a series for the Bessel functions.

Definition. A *linear* ODE of order n, in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in, the form

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = G(x)$$

where $P_n(x)$ is not identically 0. The term $G(x)$ is called the *nonhomogeneous term.* If $G(x)$ is identically 0, then the DE is said to be *homogeneous.*

eg. Bessel's equation is $x^2y''+xy'+(x^2-p^2)y = 0$ is a linear homogeneous DE.

eg. The Pendulum Equation is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

where θ is the angle of displacement from the vertical, g is the gravitational constant, l is the length of the pendulum and t is time. This is a nonlinear DE of order 2.

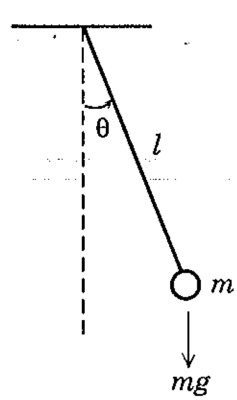


FIGURE 1.1.1 An oscillating pendulum.

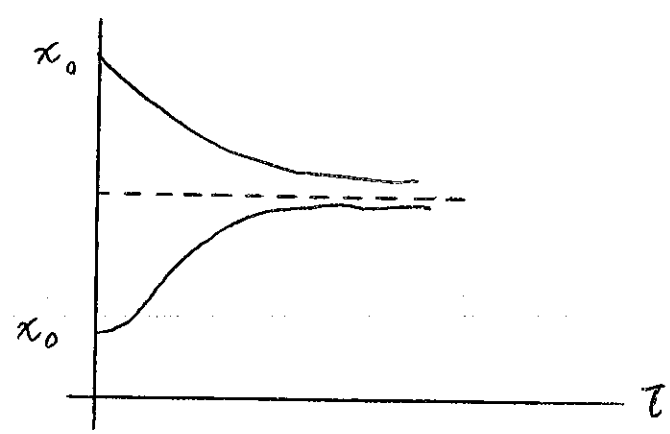
Definition. Consider the n^{th} order ODE $F[x, y, y', \ldots, y^{(n)}] = 0$ where F is a real function of its $n + 2$ arguments $x, y, y', \ldots, y^{(n)}$.
(i) Let f be a real function defined for all x in an interval I and having an n^{th} derivative for all $x \in I$. f is called an *explicit solution* of the above DE on I if $F[x, f, f', \ldots, f^{(n)}] = 0$ for all $x \in I$.
(ii) A relation $g(x, y) = 0$ is an *implicit solution* of the DE if it defines (implicitly) a function f of x on I that is an explicit solution to the DE.

eg. The DE $y' + \dfrac{x}{y} = 0$ has solutions implicitly defined by $x^2 + y^2 = k$.

eg. The *Logistic Equation* is $x' = ax(k - x)$. This is a first order nonlinear DE. An explicit solution can be found by integration (and partial fractions):

$$x = \frac{akx_0}{ax_0 + (ak - ax_0)e^{-at}}.$$

This equation represents the growth of a population with *carrying capacity* k and initial population size x_0.



Note. An n^{th} order linear DE (under the "appropriate conditions") will have an ***n-parameter family of solutions.***

eg. The DE $y'' + y = 0$ has the 2-parameter family of solutions

$$y = c_1 \cos(x) + c_2 \sin(x).$$

Definition. A DE can have "supplementary conditions" on the solution. If all the supplementary conditions relate to one value of the independent variable x, then the problem of finding a solution to the DE satisfying the supplementary conditions is called an ***initial value problem*** (or "IVP" for short). If the supplementary conditions relate to 2 or more x values, the problem is called a ***boundary value problem*** (or "BVP" for short).

eg. Solve the IVP:

$$\begin{aligned} y'' + y &= 0 \\ y(0) &= 1 \\ y'(0) &= 2. \end{aligned}$$

The solution is: $y = \cos x + 2\sin x$.

eg. Solve the BVP:

$$\begin{aligned} y'' + y &= 0 \\ y(0) &= 1 \\ y(\pi/2) &= 2. \end{aligned}$$

The solution is: $y = \cos x + 2 \sin x$.

Note. So, the DE $y'' + y = 0$ has a 2-parameter family of solutions and the 2-parameters can be determined if 2 supplementary conditions are given.

Existence Theorems for Linear ODEs

Definition. The n functions $f_1, f_2, \ldots, f_n$ are called *linearly dependent* on $x \in [a, b]$ if there exists constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x such that $x \in [a, b]$. n functions are called *linearly independent* on the interval $x \in [a, b]$ if they are not linearly dependent there.

Theorem. Basic Existence and Uniqueness Theorem for Homogeneous Linear ODEs.
The n^{th} order homogeneous linear DE

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = 0$$

where $P_0, P_1, \ldots, P_n$ are continuous for $x \in [a, b]$ and $P_n(x) \neq 0$ for $x \in [a, b]$ has n linearly independent solutions. Also, if $f_1, f_2, \ldots, f_n$ are linearly independent solutions of the DE then any linear combination $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ is also a solution (another reason to call these DEs "linear"!). Additionally, any solution of the DE can be written as some linear combination of the f_i's.

Theorem. Basic Existence and Uniqueness Theorem for Nonhomogeneous Linear ODEs.
Let y_p be a *particular solution* of the nonhomogeneous linear DE

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = G(x)$$

and let y_c be the general solution (also called the *complementary function*) of the associated linear homogenous DE

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = 0.$$

Then every solution of the original nonhomogeneous DE is of the form

$$y = y_c + y_p.$$

eg. Solve $y'' + y = e^x$. Notice that $y_p = e^x/2$.
The general solution is $y = c_1 \cos x + c_2 \sin x + e^x/2$.

Systems of Linear ODEs

Note. We sift notation a bit now. We will let t be the independent variable and x the dependent variable.

Definition. A *system* of n first order linear DEs is something of the form:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1,n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2,n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{n,n}(t)x_n + g_n(t) \end{aligned}$$

Notice that we can write such a system in matrix form:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t).$$

We consider the case where the $\mathbf{P}(t)$ is a constant matrix and $\mathbf{g}(t) = \mathbf{0}$.

eg. It is rather easy to see that the solution to the first order homogeneous linear IVP

$$\begin{aligned} x' &= ax \\ x(0) &= x_0 \end{aligned}$$

is $x = x_0 e^{at}$.

eg. In light of the previous example, when considering the linear homogeneous first order system with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we seek a soultion of the form $\mathbf{x} = \mathbf{a}e^{rt}$ where $\mathbf{a}$ is a constant vector. If this is a solution, then

$$\begin{aligned} \mathbf{x}' &= \mathbf{A}\mathbf{x} \\ \mathbf{a}re^{rt} &= \mathbf{A}\mathbf{a}e^{rt} \\ r\mathbf{a} &= \mathbf{A}\mathbf{a} \\ \mathbf{A}\mathbf{a} - r\mathbf{a} &= \mathbf{0} \\ (\mathbf{A} - r\mathbf{I})\mathbf{a} &= \mathbf{0} \end{aligned}$$

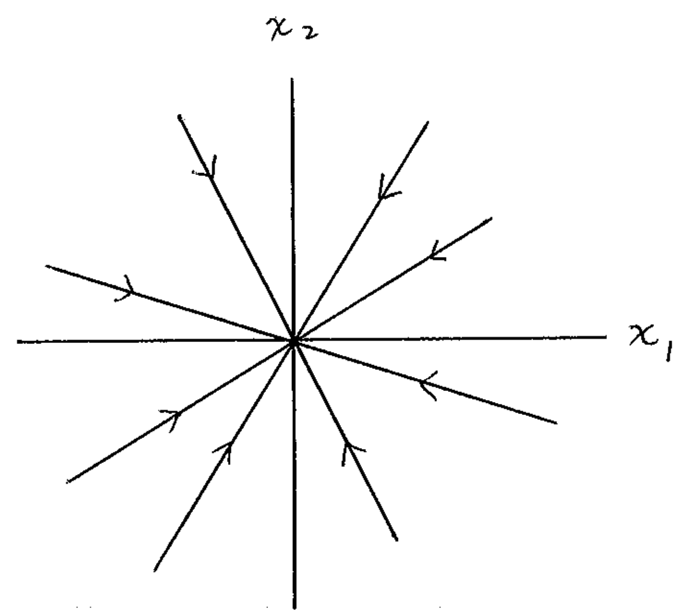
So, we have such a solution if r is an eigenvalue of $\mathbf{A}$ and $\mathbf{a}$ is an eigenvector of $\mathbf{A}$.

eg. Consider $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$. The eigenvalues of the coefficient matrix $\mathbf{A}$ are -1 and -2. An eigenvector associated with -1 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and an eigenvector associated with -2 is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. So the general solution of the system of DEs is $\mathbf{x} = c_1 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t} \\ 3e^{-2t} \end{pmatrix}$.

The Phase Plane

Definition. If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is any particular solution of a system of DEs (determined by the values of c_1 and c_2, which are derived from initial or boundary values), then we can represent this solution as a *trajectory* in the x_1x_2-plane (called the *phase plane*).

eg. For the previous example, if we let $\mathbf{x}^{(1)} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$ and $\mathbf{x}^{(2)} = \begin{pmatrix} e^{-2t} \\ 3e^{-2t} \end{pmatrix}$ then the general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$. In the phase plane, we have the following trajectories:

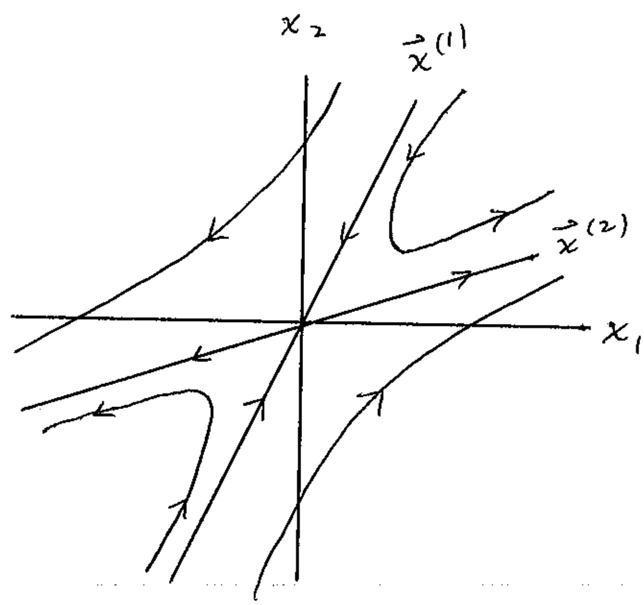


Note. If $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a symmetric matrix, then the eigenvalues of $\mathbf{A}$ are real and the eigenvectors are linearly independent. In this case, the general solution to this DE is

$$\mathbf{x} = c_1\mathbf{a}^{(1)}e^{r_1 t} + c_2\mathbf{a}^{(2)}e^{r_2 t} + \cdots + c_n\mathbf{a}^{(n)}e^{r_n t}$$

where $r_1, r_2, \ldots, r_n$ are the eigenvalues of $\mathbf{A}$ and $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ are the respective eigenvectors.

eg. Consider $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$. The eigenvalues of the coefficient matrix $\mathbf{A}$ are -1 and 2 with corresponding eigenvectors of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. So the general solution of the system of DEs is $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t}$. In the phase plane:



Note. Cauchy's formula is $e^{x+iy} = e^x e^{-iy} = e^x(\cos y + i \sin y)$. Applying this we get:

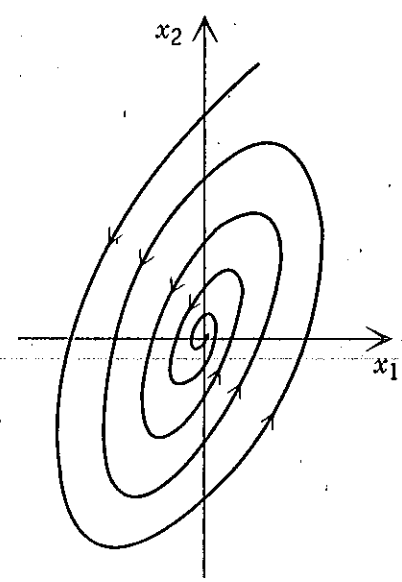
Theorem. If $\mathbf{A}$ is real, and $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ are eigenvalues of $\mathbf{A}$ with corresponding eigenvectors $\mathbf{a}^{(1)} = \mathbf{a} - i\mathbf{b}$ and $\mathbf{a}^{(2)} = \mathbf{a} - i\mathbf{b}$, then two linearly independent solutions to $\mathbf{x}' = \mathbf{Ax}$ are

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)) \text{ and}$$
$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\cos(\mu t) + \mathbf{b}\sin(\mu t)).$$

eg. Consider $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$. The eigenvalues of $\mathbf{A}$ are $-1 \pm 2i$. From the previous theorem, we get that the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -2e^{-t}\sin 2t \\ e^{-t}\cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{pmatrix}.$$

In the phase plane, the trajectories are:

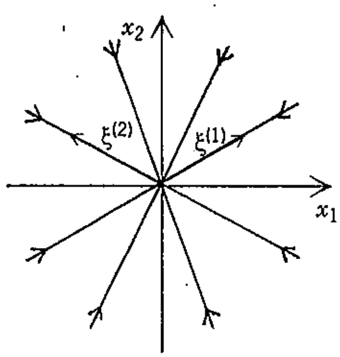


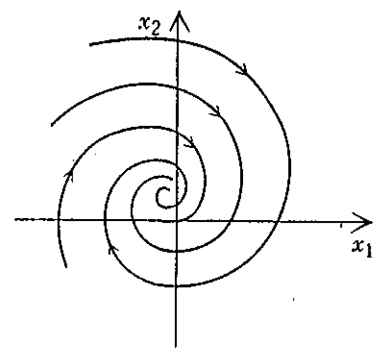
Critical Points and Stability

Definition. The system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has a *critical point* at $\mathbf{x} = \mathbf{x}_0$ if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$. Notice that the system is in equilibrium at a critical point.

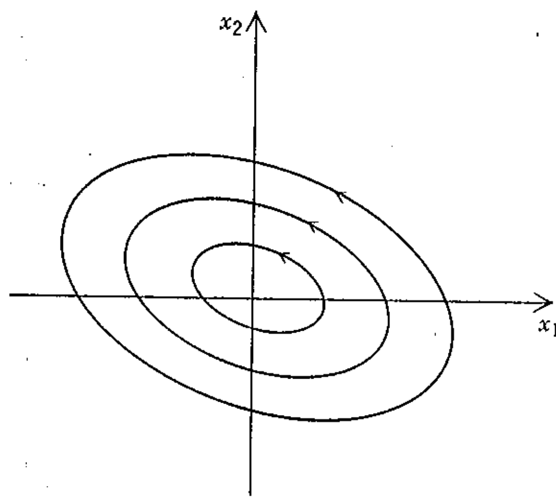
Note. We can classify the critical points of a system of DEs as
(i) *asymptotically stable*: all solutions approach $\mathbf{0}$ as $t \to \infty$,
(ii) *stable*: all solutions remain bounded and do not approach $\mathbf{0}$ as $t \to \infty$, or
(iii) *unstable*: some trajectories approach ∞ as $t \to \infty$.

(i)

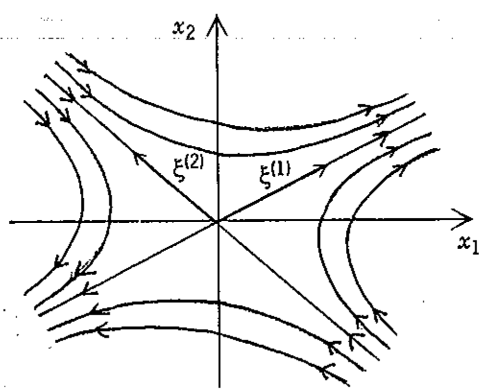


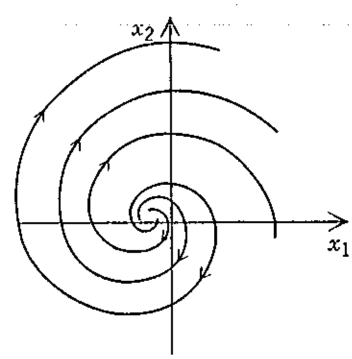


(ii)



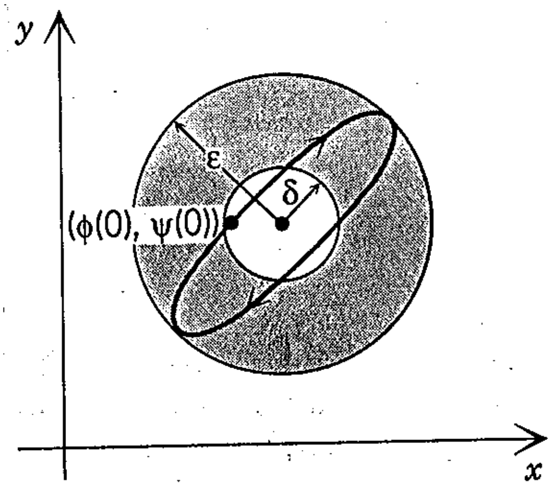
(iii)



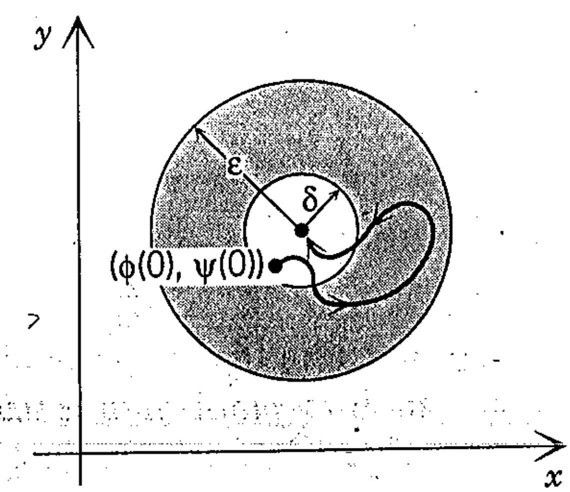


Definition. The system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is *autonomous* ($\mathbf{x}'$ does not explicitly depend on t, only on $\mathbf{x}$).

Definition. A critical point $\mathbf{x}^0$ of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is said to be *stable* is for all $\epsilon > 0$ there exists a $\delta > 0$ such that every solution $\mathbf{x} = \mathbf{\Phi}(t)$ which at $t = t_0$ satisfies $\|\mathbf{\Phi}(t_o) - \mathbf{x}^0\| < \delta$, exists for all $t \geq t_0$ and satisfies $\|\mathbf{\Phi}(t) - \mathbf{x}^0\| < \epsilon$ for all $t \geq t_0$.



Definition. A critical point $\mathbf{x}^0$ is said to be *asymptotically stable* if it is stable and there exists a $\delta_0 > 0$ such that if $\|\mathbf{\Phi}(t_0) = \mathbf{x}^0\| < \delta_0$ then $\lim_{t\to\infty} \mathbf{\Phi}(t) = \mathbf{x}^0$.



The Pendulum Equation
Damped and Undamped

eg. Recall that the pendulum equation was

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

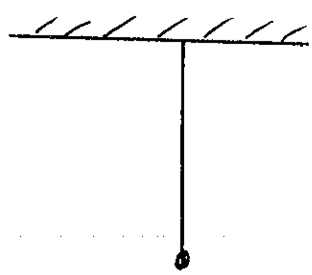
We add a further complication by assuming that there is a damping proportional to linear velocity. Then we have

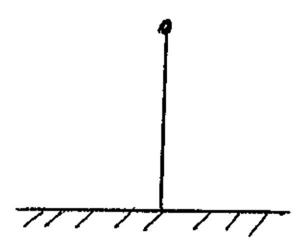
$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - \frac{c}{ml}\frac{d\theta}{dt}.$$

We can write this DE as a system by letting $x = \theta$ and

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \frac{-g}{l}\sin x - \frac{c}{ml}y.\end{aligned}$$

There is a critical point at $y = 0$ and $x = n\pi$ where n is an integer. These critical points correspond to two equilibria:





The system can be written as

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ \frac{-g}{l} & \frac{-c}{ml} \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ \frac{-g}{l}(\sin x - x) \end{pmatrix}.$$

Near critical points, the second term on the right hand side is "relatively unimportant." This means that we can study stability of the critical points by looking at the eigenvalues of the **A** matrix. We find that there is asymptotic stability at $x = \theta = 2n\pi$ and instability at $x = \theta = (2n+1)\pi$.

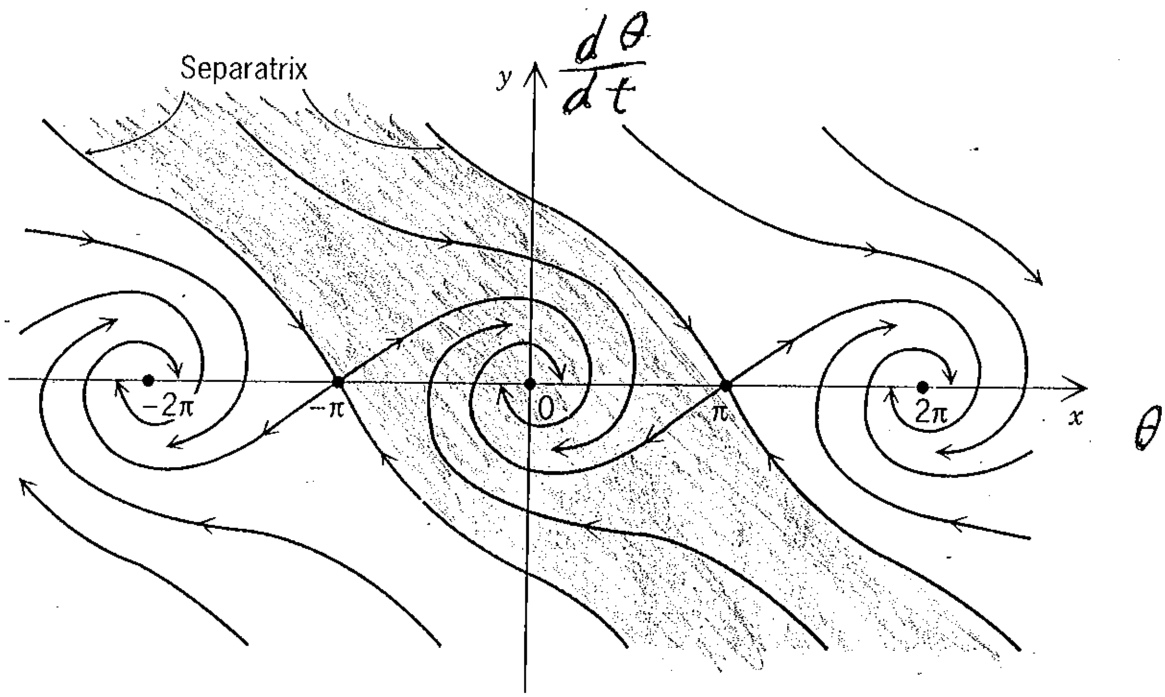


FIGURE 9.3.5 Phase portrait for the damped pendulum.

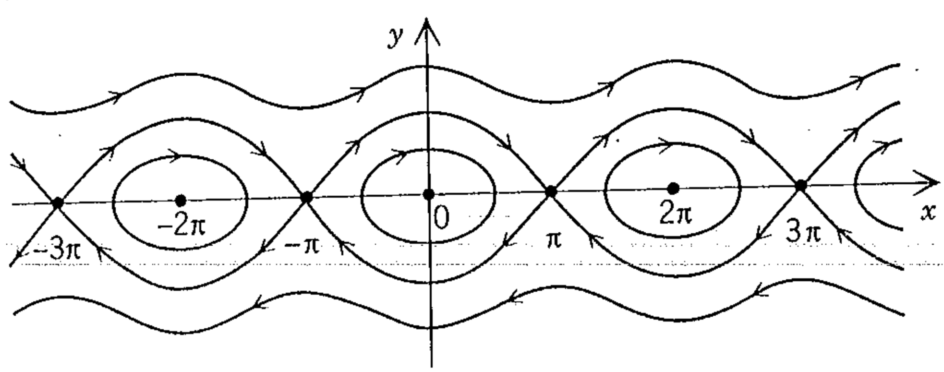
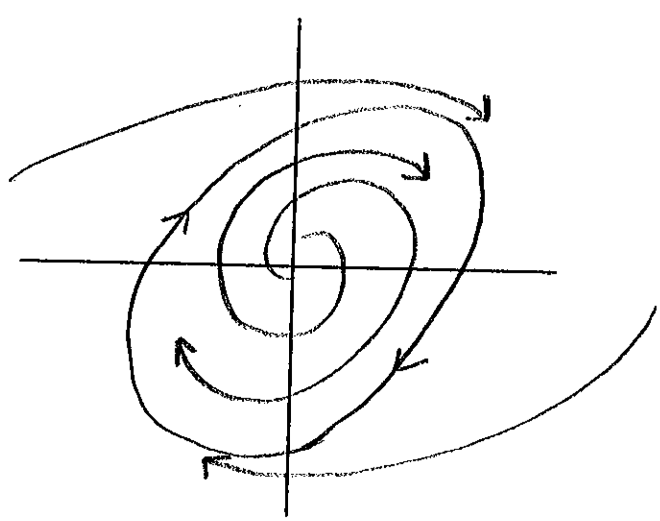


FIGURE 9.3.6 Phase portrait for the undamped pendulum.

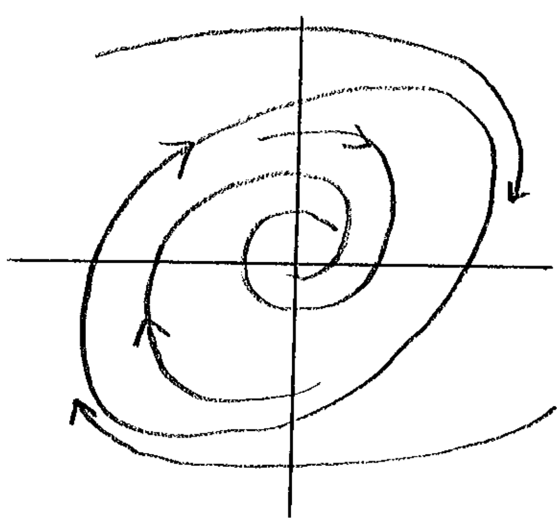
Limit Cycles and the Poincare-Bendixson Theorem

Definition. A solution to the autonomous system $\mathbf{x}' = f(\mathbf{x})$ is *periodic* if for some constant T, $\mathbf{x}(t+T) = \mathbf{x}(t)$. Trajectories of periodic solutions are simple closed curves in the phase plane.

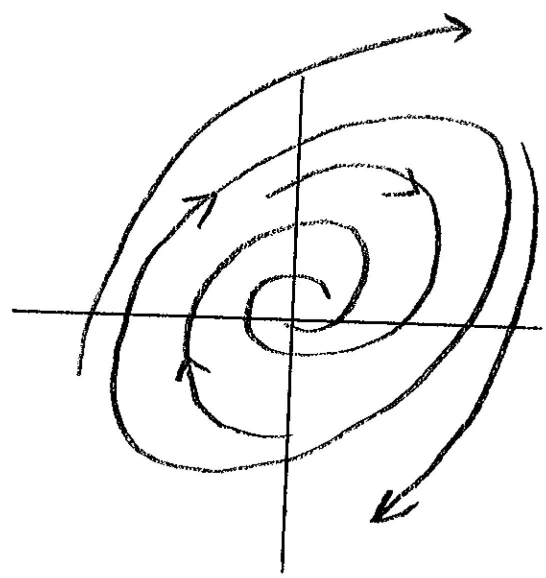
Definition. A closed trajectory in the phase plane such that other trajectories spiral toward it (either from the inside or outside) as $t \to \infty$ is called a *limit cycle.*



Definition. If all the trajectories near a limit cycle (both those inside and outside) spiral towards the limit cycle as $t \to \infty$ then the limit cycle is said to be ***stable***. If the trajectories on one side spiral towards and on the other side spiral away, then the limit cycle is ***semistable***. If the trajectories on both sides of a closed trajectory spiral away as $t \to \infty$, then the closed trajectory is ***unstable*** (in fact, it isn't even called a "limit cycle" in this case). In the case that nearby trajectories neither approach nor depart a closed trajectory, it is said to be ***neutrally stable.***



SEMISTABLE



UNSTABLE

Theorem. Suppose

$$\mathbf{x}' = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$$

where F and G have continuous first partial derivatives in a domain D of the xy−plane. A closed trajectory must necessarily enclose at least one critical point. If it encloses only one critical point, then the critical point cannot be a "saddle point."

Theorem. Poincaré-Bendixson

Let F and G have continuous first partial derivatives in a domain (i.e. open connected set) D of the xy−plane. Let D_1 be a bounded subdomain in D and let R be the region that consists of D_1 and its boundary. Suppose that R contains no critical point. If $x = \Phi(t)$ and $y = \Psi(t)$ is a solution for all $t \geq t_0$ and the points $(\Phi(t), \Psi(t))$ are in R for $t \geq t_0$. Then either

(i) $x = \Phi(t)$, $y = \Psi(t)$ is a periodic solution, or

(ii) $x = \Phi(t)$, $y = \Psi(t)$ spirals towards a closed trajectory as $t \to \infty$.

In either case, R contains a periodic solution.

Some Funny Stuff from the Third Dimension

Note. In the last several years, it has come to the attention of the mathematical community that third and higher order systems can exhibit very complex behavior not seen in second order systems. In the early 1960's, Edward Lorenz was running a meteorological model on his computer. He was numerically solving the equations:

$$\begin{aligned}\frac{dx}{dt} &= \sigma(-x+y)\\ \frac{dy}{dt} &= rx - y - xz\\ \frac{dz}{dt} &= -bz + xy\end{aligned}$$

The system has the critical points:

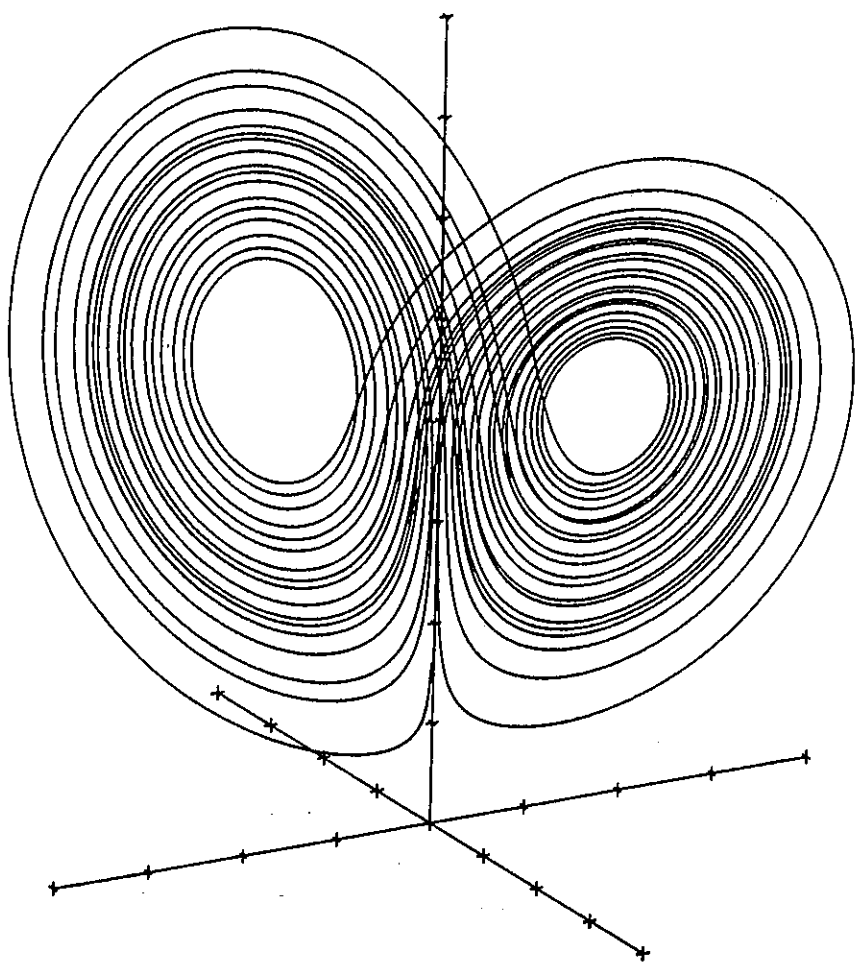
$$\begin{aligned}P_1 &= (0,0,0),\\ P_2 &= (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1), \text{ and}\\ P_3 &= (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1).\end{aligned}$$

If $r \leq 1$, then there is only one critical point and we find that it is asymptotically and globally stable. If $r > 1$ then $(0,0,0)$ is unstable.

If $1 < r < 1.34$, then P_2 and P_3 are asymptotically stable. Solutions asymptotically approach P_2 or P_3 (not spiralling).

If $1.34 < r < 24.74$, then P_2 and P_3 are asymptotically stable and solutions spiral toward P_2 or P_3.

If $r > 24.74$, then all critical points are unstable. It can be shown that trajectories remain bounded as $t \to \infty$. In fact, trajectories approach a limiting set of zero volume. This attracting set is called a *strange attractor*!



Note. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= -my + nxz \\ \frac{dy}{dt} &= mx + nyz \\ \frac{dz}{dt} &= (n/2)(1 + z^2 - x^2 - y^2).\end{aligned}$$

If m and n are relatively prime, then trajectories are torus knots of type (m, n). In particular, with $m = 2$ and $n = 3$, trajectories are the trefoil knot: