The Quaternions: An Algebraic and Analytic Exploration

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Available online: http://faculty.etsu.edu/gardnerr/talks/Quaternions-Auburn-Beamer.pdf

Ring

Definition. [Hungerford Definition III.1.1] A *ring* is a nonempty set R together with two binary operations (denoted $+$ and multiplication) such that:

(i) $(R, +)$ is an abelian group.

(ii) $(ab)c = a(bc)$ for all $a, b, c \in R$ (i.e., multiplication is associative).

(iii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ (left and right distribution of multiplication over $+$).

If in addition,

(iv) $ab = ba$ for all $a, b \in R$,

then R is a *commutative ring*. If R contains an element 1_R such that

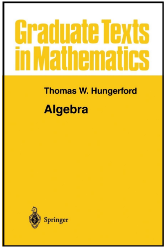
(v) $1_R a = a 1_R = a$ for all $a \in R$,

then R is a *ring with identity* (or *unity*).

Zero Divisors

Note. An obvious "shortcoming" of rings is the possible absence of inverses under multiplication. We adopt the standard notation from $(R, +)$. We denote the $+$ identity as 0 and for $n \in \mathbb{Z}$ and $a \in R$, na denotes the obvious repeated addition.

Definition. [Hungerford Definition III.1.3] A nonzero element a in the ring R is a *left* (respectively, *right*) *zero divisor* if there exists a nonzero $b \in R$ such that $ab = 0$ (respectively, $ba = 0$). A *zero divisor* is an element of R which is both a left and right zero divisor.



The Quaternions, $\mathbb{H}$

Definition. [Hungerford page 117] Let $S = \{1, i, j, k\}$. Let $\mathbb{H}$ be the additive abelian group $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and write the elements of $\mathbb{H}$ as formal sums $(a_0, a_1, a_2, a_3) = a_0 1 + a_1 i + a_2 j + a_3 k$. Addition in $\mathbb{H}$ is as expected: $(a_0 + a_1 i + a_2 j + a_3 k) + (b_0 + b_1 i + b_2 j + b_3 k)$

$$= (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k.$$

We turn $\mathbb{H}$ into a ring by defining multiplication as

$$(a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) = (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3)$$
$$+(a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2)i + (a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3)j$$
$$+(a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1)k.$$

This product can be interpreted by considering:

(i) multiplication in the formal sum is associative,

(ii) $ri = ir$, $rj = jr$, $rk = kr$ for all $r \in \mathbb{R}$,

(iii) $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

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Division Ring

Definition. [Hungerford Definition III.1.5] A commutative ring R with (multiplicative) identity 1_R and no zero divisors is an *integral domain*. A ring D with identity $1_D \neq 0$ in which every nonzero element is a unit is a *division ring*. A *field* is a commutative division ring.

Note. First, it is straightforward to show that $1 = (1, 0, 0, 0)$ is the identity in $\mathbb{H}$. However, since $ij = -ji \neq ji$, then $\mathbb{H}$ is not commutative and so $\mathbb{H}$ is not an integral domain nor a field.

Noncommutative Division Ring

Theorem. The quaternions form a noncommutative division ring.

Proof. Tedious computations confirm that multiplication is associative and the distribution law holds. We now show that every nonzero element of $\mathbb{H}$ has a multiplicative inverse. Consider $q = a_0 + a_1 i + a_2 j + a_3 k$. Define $d = a_0^2 + a_1^2 + a_2^2 + a_3^2 \neq 0$. Notice that

$$\begin{aligned}
&(a_0 + a_1 i + a_2 j + a_3 k)((a_0/d) - (a_1/d)i - (a_2/d)j - (a_3/d)k) \\
&= (a_0(a_0/d) - a_1(-a_1/d) - a_2(-a_2/d) - a_3(-a_3/d)) \\
&+(a_0(-a_1/d) + a_1(a_0/d) + a_2(-a_3/d) - a_3(-a_2/d))i \\
&+(a_0(-a_2/d) + a_2(a_0/d) + a_3(-a_1/d) - a_1(-a_3/d))j \\
&+(a_0(-a_3/d) + a_3(a_0/d) + a_1(-a_2/d) - a_2(-a_1/d))k \\
&= (a_0^2 + a_1^2 + a_2^2 + a_3^2)/d = 1.
\end{aligned}$$

So $(a_0 + a_1 i + a_2 j + a_3 k)^{-1} = (a_0/d) - (a_1/d)i - (a_2/d)j - (a_3/d)k$ where $d = a_0^2 + a_1^2 + a_2^2 + a_3^2$. Therefore every nonzero element of $\mathbb{H}$ is a unit and so the quaternions form a noncommutative division ring. □

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&= (a_0(a_0/d) - a_1(-a_1/d) - a_2(-a_2/d) - a_3(-a_3/d)) \\
&+(a_0(-a_1/d) + a_1(a_0/d) + a_2(-a_3/d) - a_3(-a_2/d))i \\
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The Quaternion Group of Order 8

Note. Since every nonzero element of $\mathbb{H}$ is a unit, the $\mathbb{H}$ contains no left zero divisors: If $qq' = 0$ and $q \neq 0$, then $q' = q^{-1}0 = 0$. Similarly, $\mathbb{H}$ has no right zero divisors.

The order 8 *group of quaternions* has multiplication table:

$\cdot$	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	$-j$	$-i$	1	$-k$	j
j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

Notice that each of i, j, and k are square roots of -1. So the quaternions are, in a sense, a generalization of the complex numbers $\mathbb{C}$.

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j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

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Matrix Representation

Note. The quaternions may also be interpreted as a subring of the ring of all 2×2 matrices over $\mathbb{C}$. This is Exercise III.1.8 of Hungerford: "Let R be the set of all 2×2 matrices over the complex field $\mathbb{C}$ of the form $\begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}$, where $\overline{z}, \overline{w}$ are the complex conjugates of z and w, respectively. Prove that R is a division ring and that R is isomorphic to the division ring of real quaternions." In fact, the quaternion group, Q_8, can be thought of as the group of order 8 generated by $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, under matrix multiplication.

Note. In fact, the complex numbers can be similarly represented as the field of all 2×2 matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$.

2 and 3 Dimensional Number Systems

Note. The complex numbers can be defined as ordered pairs of real numbers, $\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$, with addition defined as $(a, b) = (c, d) = (a + c, b + d)$ and multiplication defined as $(a, b)(c, d) = (ac - bd, bc + ad)$. We then have that $\mathbb{C}$ is a field with additive identity $(0, 0)$ and multiplicative identity $(1, 0)$. The additive inverse of (a, b) is $(-a, -b)$ and the multiplicative inverse of $(a, b) \neq (0, 0)$ is $(a/(a^2 + b^2), -b/(a^2 + b^2))$. We commonly denote (a, b) as "$a + ib$" so that $i = (0, 1)$ and we notice that $i^2 = -1$. The complex numbers are visualized as the "complex plane" where $a + ib \in \mathbb{C}$ is associated with $(a, b) \in \mathbb{R}^2$. During the early decades of the 19th century, the complex numbers became an accepted part of mathematics (in large part due to the development of complex function theory by Augustin Cauchy). Since the complex numbers have an interpretation as a sort of "two dimensional" number system, a natural question to ask is: "Is there a three (or higher) dimensional number system?"

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William Rowan Hamilton

Note. Sir William Rowan Hamilton (1805–1865) spent the years 1835 to 1843 trying to develop a three dimensional number system based on triples of real numbers. He never succeeded. However, he did succeed in developing a four dimensional number system, now called the quaternions and denoted "$\mathbb{H}$" in his honor. In a letter he wrote late in his life to his son Archibald Henry, Hamilton tells the story of his discovery:

> "Every morning in the early part of [October 1843], on my coming down to breakfast, your little brother, William Edwin, and yourself, used to ask me, 'Well, papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head: 'No, I can only add and subtract them.' But on the 16th day of that same month. . . An electric circuit seemed to close; and a spark flashed forth the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work by myself, if spared, and, at all events, on the part of others if I should even be allowed to live long enough distinctly to communicate the discovery. . . .

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Hamilton (continued)

> Nor could I resist the impulse—unphilosophical as it may have been—to cut with a knife on a stone of Brougham Bridge [in Dublin, Ireland; now called "Broom Bridge"], as we passed it, the fundamental formula with the symbols i, j, k:
>
> $$i^2 = j^2 = k^2 = ijk = -1$$
>
> which contains the Solution of the Problem, but, of course, the inscription has long wince mouldered away."

So the exact date of the birth of the quaternions is October 16, 1843 [Derbyshire 2006].

A Polynomial with No Roots

Note. We want to explore the properties of the roots of polynomials of a quaternionic variable. A general Fundamental Theorem of Algebra does not hold, as revealed by considering the polynomial $aq^n - q^n a + 1$ where a is a nonreal quaternion. With $a = a_0 + ia_1 + ja_2 + ka_3$ and $q^n = q_0 + iq_1 + jq_2 + kq_3$, where $a_0, a_1, a_2, a_3, q_0, q_1, q_2, q_3 \in \mathbb{R}$, we have

$$\begin{aligned} aq^n - q^n a + 1 &= (a_0 + ia_1 + ja_2 + ka_3)(q_0 + iq_1 + jq_2 + kq_3) \\ &\quad -(q_0 + iq_1 + jq_2 + kq_3)(a_0 + ia_1 + ja_2 + ka_3) + 1 \\ &= (a_0q_0 + ia_0q_1 + ja_0q_2 + ka_0q_3 - a_1q_1 + ia_1q_0 - ja_1q_3 + ka_1q_2 \\ &\quad -a_2q_2 + ia_2q_3 + ja_2q_0 - ka_2q_1 - a_3q_3 - ia_3q_2 + ja_3q_1 + ka_3q_0) \\ &\quad -(a_0q_0 + ia_1q_0 + ja_2q_0 + ka_3q_0 - a_1q_1 + ia_0q_1 - ja_3q_1 + ka_2q_1 \\ &\quad -a_2q_2 + ia_3q_2 + ja_0q_2 - ka_1q_2 - a_3q_3 - ia_2q_3 + ja_1q_3 + ka_0q_3) + 1 \\ &= 1 + i(2a_2q_3 - 2a_3q_2) + j(-2a_1q_3 + 2a_3q_1) + k(2a_1q_2 - 2a_2q_1) \neq 0. \end{aligned}$$

Notice we have computed the commutator

$$\begin{aligned} &[a_0 + ia_1 + ja_2 + ka_3, q_0 + iq_1 + jq_2 + kq_3] \\ &= i2(a_2q_3 - a_3q_2) + j2(a_3q_1 - a_1q_3) + k2(a_1q_2 - a_2q_1). \end{aligned}$$

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The Factor Theorem

Note. You are probably familiar with the Factor Theorem which relates roots of a polynomial to linear factor of the polynomial. You might not recall that it requires commutativity, though:
The Factor Theorem. [Hungerford Theorem III.6.6] Let R be a commutative ring with identity and $f \in R[x]$. Then $c \in R$ is a root of f if and only if $x - c$ divides f.

Note. The Factor Theorem is used to prove the following, which might remind you of the Fundamental Theorem of Algebra:
Theorem. [Hungerford Theorem III.6.7] If D is an integral domain contained in an integral domain E and $f \in D[x]$ has degree n, then f has at most n distinct roots in E.
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More Roots than Degree

Note. It is easy to see that the polynomial $q^2 + 1 \in \mathbb{H}[q]$ has more than two roots. Along with $\pm i$ are the roots $\pm j$ and $\pm k$. In fact, the polynomial has an *infinite* number of roots in $\mathbb{H}$! Let $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1^2 + x_2^2 + x_3^2 = 1$. Then

$$\begin{aligned}
(x_1 i + x_2 j + x_3 k)^2 &= x_1^2 i^2 + x_1 x_2 ij + x_1 x_3 ik + x_2 x_1 ji + x_2^2 j^2 + x_2 x_3 jk \\
&\quad + x_3 x_1 ki + x_3 x_2 kj + x_3^2 k^2 \\
&\quad \text{by the definition of multiplication} \\
&= -x_1^2 - x_2^2 - x_3^2 \text{ since } ij = -ji, ik = -ki, jk = -kj \\
&= -1 \text{ since } x_1^2 + x_2^2 + x_3^2 = 1.
\end{aligned}$$

The Sphere $\mathbb{S}$

Note. We now turn our attention to polynomials in $\mathbb{H}[x]$. We are particularly interested in roots of such polynomials, a version of the Factor Theorem, and the concept of algebraic closure. Much of this material is of fairly recent origins. The remainder of this presentation is mostly based on the following references:

1. T. Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics #131, Springer-Verlag (1991).
2. G. Gentili and D. C. Struppa, A New Theory of Regular Functions of a Quaternionic Variable, *Advances in Mathematics* **216** (2007), 279–301.

Definition. We denote by $\mathbb{S}$ the two dimensional sphere (as a subset of the four dimensional quaternions $\mathbb{H}$)
$\mathbb{S} = \{q = x_1 i + x_2 j + x_3 k \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. As observed above, for any $I \in \mathbb{S}$ we have $I^2 = -1$. For $x, y \in \mathbb{R}$ we let $x + y\mathbb{S}$ denote the two dimensional sphere $x + y\mathbb{S} = \{x + yI \mid I \in \mathbb{S}\}$.

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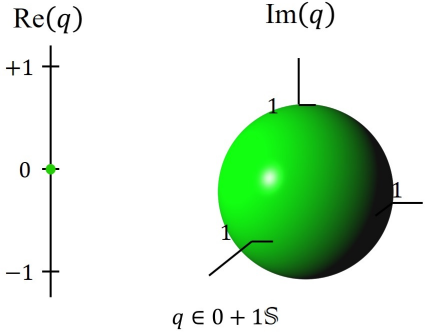
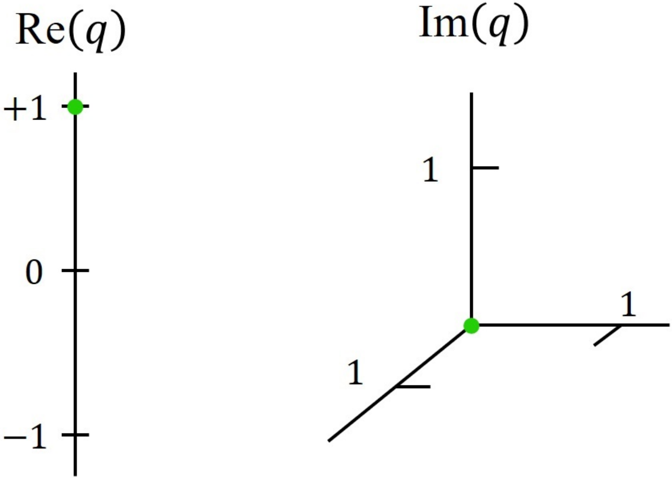
Real, Imaginary, and Vector Parts

Note. In fact, every quaternion q can be written in the form $q = x + yI$ for unique $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$. Some then denote $\text{Re}(q) = x$ and $\text{Im}(q) = y$ [Vlacci 2017], while others denote $\text{Re}(q) = x$ and $\text{Im}(q) = yI$ [Xu 2018]; in addition, for $q = x_0 + ix_1 + jx_2 + kx_3$, a common terminology is to refer to $ix_1 + jx_2 + kx_3$ as the *vector part* of q, denoted $\text{Vec}(q)$ [Gal and Sabadoni 2015].

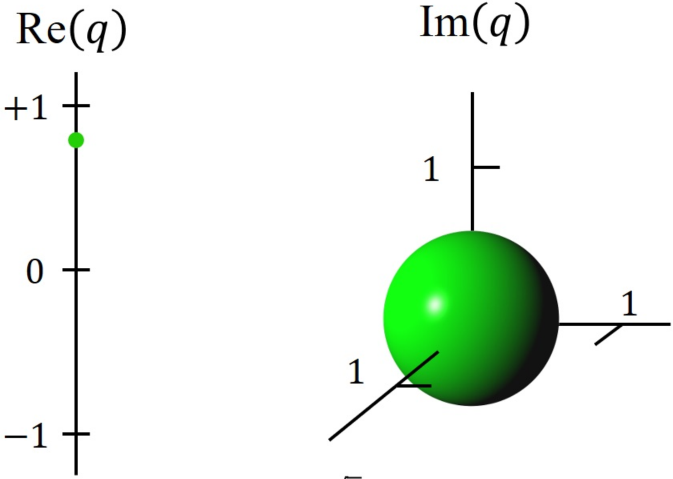


The Unit Ball



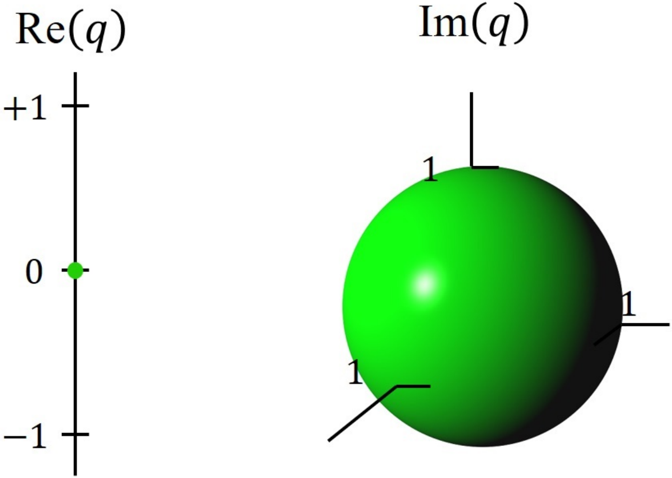
$q \in 1 + 0\mathbb{S}$

The Unit Ball



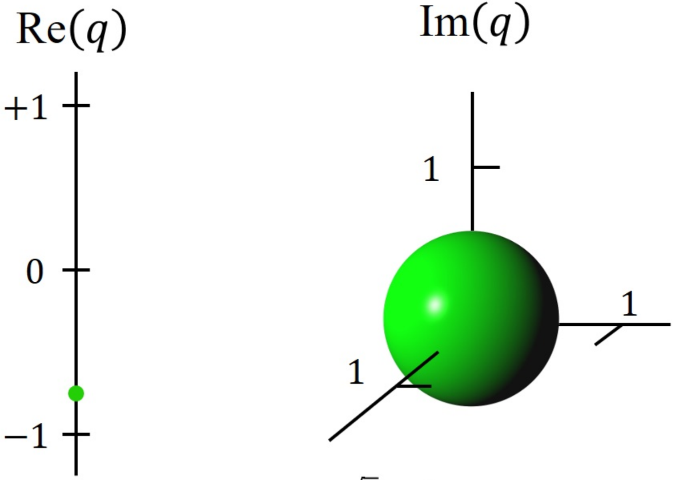
$$q \in \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbb{S}$$

The Unit Ball



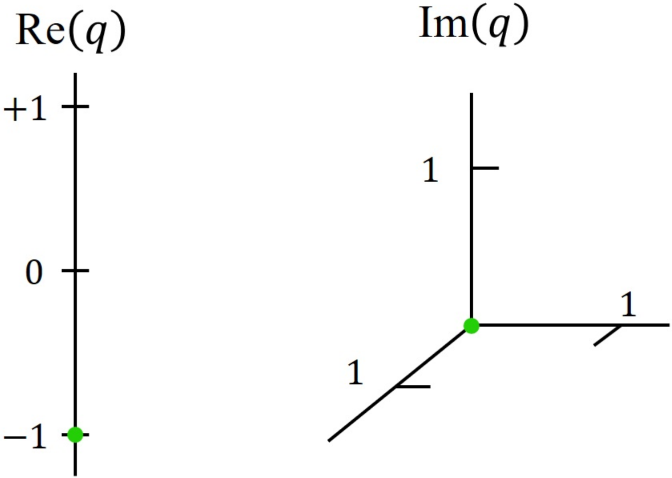
$q \in 0 + 1\mathbb{S}$

The Unit Ball



$$q \in -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathbb{S}$$

The Unit Ball



$$q \in -1 + 0\mathbb{S}$$

Polynomials and Products

Note. We take q as the indeterminate in the ring of polynomials $\mathbb{H}[q]$. Since $\mathbb{H}$ is not commutative, we are faced with the case that a monomial of the form $aq^n \in \mathbb{H}[q]$ is the same as monomial $a_0qa_1qa_2q\cdots qa_n \in \mathbb{H}[q]$ where $a = a_0a_1\cdots a_n$, but if we evaluate aq^n at some element of $\mathbb{H}$, we may get a different value than if we evaluate $a_0qa_1q\cdots qa_n$ at the same element of $\mathbb{H}$. That is, evaluation of an element of $\mathbb{H}[q]$ at $r \in \mathbb{H}$ is not a homomorphism. In the remainder of this presentation, we consider polynomials with the powers of the indeterminate on the left and the coefficients on the right: $p(q) = \sum_{i=0}^n q^i a_i$. We will call p a "quaternionic polynomial."

Definition. For two quaternionic polynomials $p_1(q) = \sum_{i=0}^n q^i a_i$ and $p_2(q) = \sum_{i=0}^m q^i b_i$ in $\mathbb{H}[q]$, define the *product* (or *regular product*)

$$(p_1p_2)(q) = \sum_{i=0,1,\ldots,n;j=0,1,\ldots,m} q^{i+j} a_i b_j.$$

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Failure of the Factor Theorem

Note. Consider the polynomial $p(q) = (q - a)(q - b) = q^2 - q(a + b) + ab$ where a and b are noncommuting quaternions. Then $q = a$ is a root of p (clearly), but $p(b) = b^2 - ba - b^2 + ab = ab - ba \neq 0$. In fact, p has exactly two roots, the other being $(\overline{b} - a)^{-1} b(\overline{b} - a)$ [Vlacci 2017]. This shows that the evaluation mapping of $\mathbb{H}[x] \to \mathbb{H}$ is not a homomorphism when we use the regular product for multiplication in $\mathbb{H}[x]$.

Note. We now explore roots of quaternionic polynomials. The following result is originally due to A. Pogorui and M. V. Shapiro (in "On the Structure of the Set of Zeros of Quaternionic Polynomials," *Complex Variables* **49**(6) (2004), 379–389) but we present an easier proof due to Gentili and Struppa in 2007.

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Zeros on a Sphere

Theorem. Let $p(q) = \sum_{n=0}^{N} q^n a_n$ be a given quaternionic polynomial. Suppose that there exist $x_0, y_0 \in \mathbb{R}$ and $I, J \in \mathbb{S}$ with $I \neq J$ such that $p(x_0 + y_0 I) = 0$ and $p(x_0 + y_0 J) = 0$. Then for all $L \in \mathbb{S}$ we have $p(x_0 + y_0 L) = 0$.

Proof. For any $n \in \mathbb{N}$ and any $L \in \mathbb{S}$ we have that $(x_0 + y_0 L)^n = \sum_{i=0}^{n} \binom{n}{i} x_0^{n-i} y_0^i L^i = \alpha_n + L\beta_n$ by the Binomial Theorem for a ring with identity (since $x_0 y_0 L = L x_0 y_0$ because $x_0, y_0 \in \mathbb{R}$; see Theorem III.1.6 of Hungerford) where

$$\alpha_n = \sum_{i \equiv 0 (\text{mod } 4)} \binom{n}{i} x_0^{n-i} y_0^i - \sum_{i \equiv 2 (\text{mod } 4)} \binom{n}{i} x_0^{n-i} y_0^i$$

and

$$\beta_n = \sum_{i \equiv 1 (\text{mod } 4)} \binom{n}{i} x_0^{n-i} y_0^i - \sum_{i \equiv 3 (\text{mod } 4)} \binom{n}{i} x_0^{n-i} y_0^i$$

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Zeros on a Sphere (continued 1)

Proof (continued). ...because $L^{0 \pmod 4} = 1$, $L^{1 \pmod 4} = L$, $L^{2 \pmod 4} = -1$, and $L^{3 \pmod 4} = -L$. We therefore have

$$0 = 0 - 0 = p(x_0 + y_0 I) - p(x_0 + y_0 J) = \sum_{n=0}^{N} (\alpha_n + I\beta_n) a_n - \sum_{n=0}^{N} (\alpha_n + J\beta_n) a_n$$

$$= \sum_{n=0}^{N} ((\alpha_n + I\beta_n) - (\alpha_n + J\beta_n)) a_n = \sum_{n=0}^{N} (I - J)\beta_n a_n = (I - J) \sum_{n=0}^{N} \beta_n a_n.$$

By hypothesis, $I - J \neq 0$ so (since $\mathbb{H}$ has no zero divisors) $\sum_{n=0}^{N} \beta_n a_n = 0$ and so

$$0 = p(x_0 + y_0 I) = \sum_{n=0}^{N} (x_0 + y_0 I)^n a_n = \sum_{n=0}^{N} (\alpha_n + I\beta_n) a_n$$

$$= \sum_{n=0}^{N} \alpha_n a_n + I \sum_{n=0}^{N} \beta_n a_n = \sum_{n=0}^{N} \alpha_n a_n.$$

Zeros on a Sphere (continued 2)

Theorem. Let $p(q) = \sum_{n=0}^{N} q^n a_n$ be a given quaternionic polynomial. Suppose that there exist $x_0, y_0 \in \mathbb{R}$ and $I, J \in \mathbb{S}$ with $I \neq J$ such that $p(x_0 + y_0 I) = 0$ and $p(x_0 + y_0 J) = 0$. Then for all $L \in \mathbb{S}$ we have $p(x_0 + y_0 L) = 0$.

Proof (continued). Now for any $L \in \mathbb{S}$ we have that

$$p(x_0 + y_0 L) = \sum_{n=0}^{N} (x_0 + y_0 L)^n a_n = \sum_{n=0}^{N} (\alpha_n + L\beta_n) a_n$$

$$= \sum_{n=0}^{N} \alpha_n a_n + L \sum_{n=0}^{N} \beta_n a_n = 0 + 0 = 0.$$

□

Note. In fact, Gentili and Struppa develop a theory of analytic functions of a quaternionic variable and show that the previous result holds for an analytic function.

Note. In a ring of polynomials, $R[t]$, each element of R commutes with indeterminate t (see Hungerford's Theorem III.5.2(ii)). So in $R[t]$ we have that $f(t) = \sum_{i=0}^{n} a_i t^i = \sum_{i=0}^{n} t^i a_i$. However, for $r \in R$ where R is not commutative we likely have $\sum_{i=0}^{n} a_i r^i \neq \sum_{i=0}^{n} r^i a_i$. So in order to evaluate $f(r)$, we must decide on a standard representation of $f(t)$. Here, we use the form $f(t) = \sum_{i=0}^{n} t^i a_i \in R[t]$. Additionally, we may have $f(t) = g(t)h(t)$ in $R[t]$, but we may not have $f(r) = g(r)h(r)$. Consider $g(t) = t - a$ and $h(t) = t - b$ where $a, b \in R$ do not commute (so $ab \neq ba$). Then we have by the definition of multiplication that $f(t) = g(t)h(t) = (t - a)(t - b) = t^2 - t(a + b) + ab$. But

$$f(a) = a^2 - a(a + b) + ab = ab - ba \neq 0 = g(a)h(a).$$

A One-Sided Factor Theorem

Definition. [Lam Definition 16.1] Let R be a ring and $f(t) = \sum_{i=0}^n t^i a_i \in R[t]$. An element $r \in R$ is a *left root* of f if $f(r) = \sum_{i=0}^n r^i a_i = 0$. If $g(t) = \sum_{i=0}^n a_i t^i \in R[t]$. An element $r \in R$ is a *right root* of g if $g(r) = \sum_{i=0}^n a_i r^i = 0$.

The Factor Theorem in a Ring with Unity. [Lam Proposition 16.2] An element $r \in R$ is a left root of a nonzero polynomial $f(t) = \sum_{i=0}^n t^i a_i \in R[t]$ if and only if $t - r$ is a left divisor of $f(t)$ in $R[t]$.

Proof. We give a proof for left roots and divisors with the proof for right being similar. First, if

$$f(t) = \sum_{i=0}^{n} t^i a_i = (t - r)\sum_{i=0}^{n-1} t^i c_i = \sum_{i=0}^{n-1} t^{i+1} c_i - \sum_{i=0}^{n-1} t^i r c_i$$

$$\text{then } f(r) = \sum_{i=0}^{n-1} r^{i+1} c_i - \sum_{i=0}^{n-1} r^{i+1} c_i = 0.$$

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$$f(t) = (t - r)g(t) + f(r) = (t - r)g(t) + 0 = (t - r)g(t).$$

So $t - r$ is a left divisor of $f(t)$. □

Note. Recall a *right ideal* of a ring R is a subring I of R such that for all $r \in R$ and $x \in I$ we have $xr \in I$ (Hungerford's Definition III.2.1). We see from the Factor Theorem in a Ring with Unity that the set of polynomials in $R[t]$ having r as a left root is precisely the right ideal $(t - r)R[t] = \{(t - r)g(t) \mid g(t) \in R[t]\}$.

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Conjugate Roots

Theorem. [Lam Proposition 16.3] Let D be a division ring and let $f(t) = h(t)g(t)$ in $D[t]$. Let $d \in D$ be such that $a = h(d) \neq 0$. Then $f(d) = h(d)g(a^{-1}da)$. In particular, if d is a left root of f but not of h then the conjugate of d, $a^{-1}da$, is a left root of g.

Proof. Let $g(t) = \sum_{i=0}^{n} t^i b_i$. Then $f(t) = h(t)g(t) = \sum_{i=0}^{n} t^i h(t) b_i$ and so

$$f(d) = \sum_{i=0}^{n} d^i h(d) b_i = \sum_{i=0}^{n} d^i a b_i = \sum_{i=0}^{n} a a^{-1} d^i a b_i$$

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If d is a left root of f but not a left root of h then, since D has no zero divisors, $a^{-1}da$ must be a left root of g. □

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Algebraic Conjugation

Note. If D is an integral domain and $p \in D[x]$ is of degree n, then p has at most n roots in D (see Hungerford's Theorem III.6.7, mentioned above). This is not the case in a division ring as illustrated by $p(q) = q^2 + 1 \in \mathbb{H}[q]$, as described above. The following result is analogous to Hungerford's Theorem III.6.7, but for division rings. It does not imply at most n roots, but roots from at most n conjugacy classes.

Note. Quaternion a is a *conjugate* of quaternion b (in the algebraic sense) if $a = cbc^{-1}$ for some quaternion c. Notice that if $a = c_1 b_1 c_1^{-1}$ and $a = c_2 b_2 c_2^{-1}$, then $b_1 = c_1^{-1} a c_1$ so that $b_1 = c_1^{-1}(c_2 b_2 c_2^{-1})c_1 = (c_1^{-1}c_2)b_2(c_1^{-1}c_2)^{-1}$. So conjugation is an equivalence relation and the conjugacy classes partition $\mathbb{H}$.

Roots in Conjugacy Classes

Theorem. [Lam Theorem 16.4, "Gordon-Motzkin"] Let D be a division ring and let f be a polynomial of degree n in $D[t]$. Then the left (right) roots of f lie in at most n conjugacy classes of D. If $f(t) = (t - a_1)(t - a_2) \cdots (t - a_n)$ where $a_1, a_2, \ldots, a_n \in D$, then any left (right) root of f is conjugate to some a_i.

Proof. We prove this using induction. In the base case, $n = 1$ and so f has exactly one left root and so the left roots lie in $n = 1$ conjugacy class. Now suppose that if a polynomial is of degree $n - 1$, then its left roots lie in at most $n - 1$ conjugacy classes. Let f be degree n and let c be a left root of f. Then by Proposition 16.2, $f(t) = (t - c)g(t)$ where g is of degree $n - 1$. Suppose $d \neq c$ is any other left root of f. Then by Proposition 16.3, d is a conjugate to a left root of $g(t)$ (in particular, $(d - c)^{-1}d(d - c) = r$ is a left root of g so $d = (d - c)r(d - c)^{-1}$).

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Proof (continued). Since by the induction hypothesis the left roots of g lie in at most $n - 1$ conjugacy classes, then this arbitrary left root of f (arbitrary except that it is not c) must lie in one of these $n - 1$ conjugacy classes. Adding in the conjugacy class containing c, we have that the left roots of f lie in at most n conjugacy classes. The result now follows in general by induction.

The proof of the second claim follows similarly by induction. The result for right roots is similar. □

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The proof of the second claim follows similarly by induction. The result for right roots is similar. □

Analytic Conjugation

Definition. For $q = a + bi + cj + dk \in \mathbb{H}$, we define the *quaternionic conjugate* $\overline{q} = a - bi - cj - dk$.

Note. For $q = a + bi + cj + dk \in \mathbb{H}$, we have

$$\begin{aligned} q\overline{q} &= (a + bi + cj + dk)(a + (-b)i + (-c)j + (-d)k) \\ &= ((a)(a) - (b)(-b) - (c)(-c) - (d)(-d)) \\ &\quad +((a)(-b) + (b)(a) + (c)(-d) - (d)(-c))i \\ &\quad +((a)(-c) + (c)(a) + (d)(-b) - (b)(-d))j \\ &\quad +((a)(-d) + (d)(a) + (b)(-c) - (c)(-b))k \\ &= a^2 + b^2 + c^2 + d^2. \end{aligned}$$

We define the *modulus* of $a \in \mathbb{H}$ as $|q| = \sqrt{q\overline{q}}$.

Conjugate of a Product

Lemma. For $q_1, q_2 \in \mathbb{H}$ we have $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$.

Proof. Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$. Then

$$
\begin{aligned}
\overline{q_1 q_2} &= \overline{(a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k)} \\
&= \overline{(a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i} \\
&\quad \overline{+(a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2)j + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)k} \\
&= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) - (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\
&\quad -(a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2)j - (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)k \\
&= ((a_2)(a_1) - (-b_2)(-b_1) - (-c_2)(-c_1) - (-d_2)(-d_1)) \\
&\quad +((-b_2)(a_1) + (-b_1)(a_2) - (-d_2)(-c_1) + (-c_2)(-d_1)i \\
&\quad +((-c_2)(a_1) + (a_2)(-c_1) - (-b_2) - d_1) + (-d_2)(-b_1))j \\
&\quad +((-d_2)(a_1) + (a_2)(-d_1) - (-c_2)(-b_1) + (-b_2)(-c_1))k \\
&\ \ \vdots
\end{aligned}
$$

Conjugate of a Product

Lemma. For $q_1, q_2 \in \mathbb{H}$ we have $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$.

Proof. Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$. Then

$$\begin{aligned}
\overline{q_1 q_2} &= \overline{(a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k)} \\
&= \overline{(a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) i} \\
&\quad \overline{+ (a_1 c_2 + c_1 a_2 + d_1 b_2 - b_1 d_2) j + (a_1 d_2 + d_1 a_2 + b_1 c_2 - c_1 b_2) k} \\
&= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) - (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) i \\
&\quad - (a_1 c_2 + c_1 a_2 + d_1 b_2 - b_1 d_2) j - (a_1 d_2 + d_1 a_2 + b_1 c_2 - c_1 b_2) k \\
&= ((a_2)(a_1) - (-b_2)(-b_1) - (-c_2)(-c_1) - (-d_2)(-d_1)) \\
&\quad + ((-b_2)(a_1) + (-b_1)(a_2) - (-d_2)(-c_1) + (-c_2)(-d_1) i \\
&\quad + ((-c_2)(a_1) + (a_2)(-c_1) - (-b_2) - d_1) + (-d_2)(-b_1)) j \\
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\end{aligned}$$

Conjugate of a Product (continued)

Lemma. For $q_1, q_2 \in \mathbb{H}$ we have $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$.

Proof (continued). ...

$$\begin{aligned}
\overline{q_1 q_2} &= ((a_2)(a_1) - (-b_2)(-b_1) - (-c_2)(-c_1) - (-d_2)(-d_1)) \\
&\quad +((a_2)(-b_1) + (-b_2)(a_1) + (-c_2)(-d_1) - (-d_2)(-c_1))i \\
&\quad +((a_2)(-c_1) + (-c_2)(a_1) + (-d_2)(-b_1) - (-b_2)(-d_1))j \\
&\quad +((a_2)(-d_1) + (-d_2)(a_1) + (-b_2)(-c_1) - (c_1)(-b_1))k \\
&= (a_2 + (-b_2)i + (-c_2)j + (-d_2)k)(a_1 + (-b_1)i + (-c_1)j + (-d_1)k) \\
&= \overline{q_2}\,\overline{q_1}.
\end{aligned}$$

□

One-Side Algebraic Closure

Note. Recall that a field is algebraically closed if every nonconstant polynomial over the field has a root in the field. This is the motivation for the following definition.

Definition. [Lam page 169] A division ring D is left (right) algebraically closed if every nonconstant polynomial in $D[t]$ has a left (right) root in D.

Note. By Proposition 16.2, if $f \in D[t]$ for left or right algebraically closed division ring D, then f can by factored into a product of linear factors in $D[t]$ (that is, f splits in $D[t]$).

Note. The following is the Fundamental Theorem of Algebra for Quaternions. The result originally appeared in I. Nivens' "Equations in Quaternions," *American Mathematical Monthly*, **48** (1941), 654–661.

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One-Sided Fundamental Theorem of Algebra

Theorem. [Lam Theorem 16.14, "Niven-Jacobson" or "Fundamental Theorem of Algebra for Quaternions"] The quaternions, $\mathbb{H}$, are left (and right) algebraically closed.

Proof. For $f(q) = \sum_{r=0}^{n} q^r a_r \in \mathbb{H}[q]$, define $\overline{f}(q) = \sum_{r=0}^{n} q^r \overline{a_r} \in \mathbb{H}[q]$. For $f, g \in \mathbb{H}[q]$ with $f(q) = \sum_{i=0}^{n} q^i a_i$ and $g(q) = \sum_{j=0}^{m} q^j b_j$ we have

$$\begin{aligned}
\overline{fg} &= \overline{\left(\sum_{i=0}^{n} q^i a_i\right)\left(\sum_{j=0}^{m} q^j b_i\right)} = \overline{\left(\sum_{i=0,1,\ldots,n;j=0,1,\ldots,m} q^{i+j} a_i b_j\right)} \\
&= \sum_{i=0,1,\ldots,n;j=0,1,\ldots,m} q^{i+j}\overline{a_i b_j} = \sum_{i=0,1,\ldots,n;j=0,1,\ldots,m} q^{i+j}\overline{b_j}\,\overline{a_i} \text{ by Lemma} \\
&= \left(\sum_{j=0}^{m} q^j \overline{b_j}\right)\left(\sum_{i=0}^{n} q^i \overline{a_i}\right) = \overline{g}\,\overline{f}.
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One-Sided Fundamental Theorem of Algebra (continued)

Proof (continued). So, in particular, $\overline{f\overline{f}} = \overline{\overline{f}}\,\overline{f} = f\overline{f}$, and so $f\overline{f}$ equals its own quaternionic conjugate. Therefore the coefficients of $f\overline{f}$ must be real and $f\overline{f} \in \mathbb{R}[q]$ for all $f \in \mathbb{H}[q]$.

We now use mathematical induction on $n = \deg(f)$ to prove that f has a left root in $\mathbb{H}$. For $n = 1$, f clearly has a left root. Suppose $n \geq 2$ and that every polynomial of degree less than n has a left root in $\mathbb{H}$. Since $\mathbb{R}(i) = \mathbb{C} \subset \mathbb{H}$ is algebraically closed and $f\overline{f} \in \mathbb{R}[q]$ then $f\overline{f}$ has a root α in $\mathbb{R}(i) = \mathbb{C}$. By Proposition 16.3, either α is a left root or f or a conjugate β of α is a left root of $\overline{f}$. In the former case we are done. In the latter case, if $f(q) = \sum_{r=0}^{n} q^r a_r$ then $\overline{f}(q) = \sum_{n=0}^{n} q^r \overline{a_r}$ and so $\overline{f}(\beta) = \sum_{r=0}^{n} \beta^r \overline{a_r} = 0$ or $\sum_{r=0}^{n} a_r \overline{\beta}^r = 0$. That is, $\overline{\beta}$ is a right root of $f(q)$. By Theorem 16.2 (applied to a right roots) we can write $f(q) = (q - \overline{\beta})g(q)$ where $g(q) \in \mathbb{H}$ has degree $n - 1$. By the induction hypothesis, $g(q)$ has a left root $\gamma \in \mathbb{H}$. But then γ is also a left root of $f(q)$ and the general result now follows by induction. The result for right algebraic closure is similar. □

One-Sided Fundamental Theorem of Algebra (continued)

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Resolution of Degree and Roots

Note. Now that we have our Fundamental Theorem of Algebra, we conclude with a brief exploration of the structure of the set of quaternions for which a polynomial has a left (right) root. The following result is from A. Pogorui and M. Shapiro's "On the Structure of the Set of Zeros of Quaternionic Polynomials," *Complex Variables: Theory and Applications* **49**(6) (2004), 379–389.

Theorem. [Pogorui and Shapiro 2004] For f a (nonzero) polynomial in $\mathbb{H}[q]$. The set of left (right) roots of f consists of isolated points or isolated two dimensional spheres of the form $S = x + y\mathbb{S}$ for $x, y \in \mathbb{R}$. The number of isolated roots plus twice the number of isolated spheres is less than or equal to n.

Resolution of Degree and Roots (continued)

Note. The proof of Pogorui and Shapiro's theorem is based on introducing a polynomial of degree $2n$ with real coefficients (called the "basic polynomial") which is associated with a given quaternionic polynomial of degree n. A one to one correspondence between the isolated zeros of the quaternionic polynomial and the basic polynomial is established, and a one to one correspondence between the isolated spheres of roots of the quaternionic polynomial and pairs of complex conjugate roots of the basic polynomial is established. Then the fact that a real polynomial of degree $2n$ has at most $2n$ complex roots (the "Fundamental Theorem of Algebra") is used to complete the proof.

Note. One would hope that Pogorui and Shapiro's theorem could be extended to an equality of the degree n and the number of isolated roots plus twice the number of isolated spheres. This would likely require an introduction of the concept of the multiplicity of a root. G. Gentili and C. Stoppato in "Zeros of Regular Functions and Polynomials of a Quaternionic Variable," *Michigan Mathematics Journal* **56** (2008), 655–667, explore exactly this. They define multiplicity (see their Definition 5.5) and give an example showing that the degree of a polynomial can exceed the sum of the multiplicities of its roots. They define the multiplicity of root p of polynomial $f(q) = \sum_{i=0}^{n} q^i a_i$ as the largest $m \in \mathbb{N}$ such that $f(q) = (q - p)^m g(q)$ where g is a polynomial (in fact, they do this for f and g quaternionic power series). They then show that $f(q) = (q - I)(q - J) = q^2 - q(I + J) + IJ$, where $I, J \in \mathbb{S}$ with $I \neq J$ and $I \neq -J$, is of degree 2 yet the only root is I which is of multiplicity 1.

Note. Pogorui and Shapiro's theorem holds if polynomial f is replaced with an analytic function of a quaternionic polynomial and the reference to the degree is dropped. This is also proved by G. Gentili and C. Stoppato (see their Theorem 2.4).

Analytic Theory 1

Note. Attempts to extend the theory of analytic functions from the complex setting to the quaternionic setting date back to at least the 1930s and pioneering work by R. Fueter [see [5] in Gentili and Struppa, 2007]. C. G. Cullen gave an alternative approach in 1965. Fueter's approach did not admit the identity function, polynomials, nor power series as regular functions; Cullen's approach admits polynomials and power series of certain forms as regular functions. Here we follow the approach of Gentili and Struppa [2007] which is based on Cullen's ideas but produces a more complete theory and allows for extensions of several results from classical complex analysis to the quaternionic setting.

Analytic Theory 2

Note. For any (fixed) $I \in \mathbb{S}$, denote the set $\mathbb{R} \oplus I\mathbb{R}$ as $\mathbb{C}_I$. Then $\mathbb{C}_I$ is isomorphic to $\mathbb{C}$ (as a field).

Definition. [Gentili and Struppa 2007, Definition 1.2] Let Ω be a domain (i.e., an open connected set) in $\mathbb{H}$. Function $f : \Omega \to \mathbb{H}$ is *C-regular* (*slice-regular* or simply *regular*) if for each $I \in \mathbb{S}$, $f|_{\mathbb{C}_I} = f_I$ is analytic on $\Omega \cap \mathbb{C}_I$.

Note. [Gentili and Struppa 2007, Remark 1.3] The requirement that $f : \Omega \to \mathbb{H}$ is regular is equivalent to the condition that for all $I \in \mathbb{S}$,

$$\overline{\partial}_I f(x + yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)[f_I(x + yI)] = 0$$

on $\Omega \cap \mathbb{C}_I$.

Analytic Theory 3

Note. Polynomials of the form studied above, $p(q) = \sum_{k=0}^{n} q^n a_n$, are regular. Since we have a metric based on modulus, $d(q, q') = |q - q'|$, then this induces a topology on $\mathbb{H}$ and we can consider sequences and series of quaternions and power series of a quaternionic variable. Fortunately, the behavior of quaternionic power series of the form $\sum_{k=0}^{\infty} q^n a_n$ are as expected.

Theorem. [Gentili and Struppa, Theorem 2.1] For every power series $\sum_{n=0}^{\infty} q^n a_n$ there exists a number $R \in \mathbb{R}_\infty$, $0 \leq R \leq \infty$, called the radius of convergence, such that the series converges absolutely for every q with $|q| < R$ and uniformly for every q with $|q| \leq \rho < R$. Moreover if $|q| > R$, the series is divergent.

Limit Points of the Set of Zeros

Theorem. [Conway Theorem IV.3.7] Let G be a connected open set of complex numbers and let $f : G \to \mathbb{C}$ be an analytic function. Then the following are equaivalent:

(a) $f \equiv 0$,

(b) there is a point a in G such that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$,

(c) $\{z \in G \mid f(z) = 0\}$ has a limit point in G.

Theorem. [Gentili and Struppa 2007, Theorem 3.1] Let $B = B(0; R) = \{q \in \mathbb{H} \mid |q| \leq R\}$ and suppose $f : B \to \mathbb{H}$ is a regular function. Denote $Z_f = \{q \in B \mid f(q) = 0\}$ the zero set of f. If there exists $I \in \mathbb{S}$ such that $L_I \cap Z_f$ has an accumulation point, then $f \equiv 0$ on B.

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Maximum Modulus Theorem

Theorem. [Conway Theorem IV.3.11] If G is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point a in G with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

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Cauchy Estimate

Theorem. [Conway, Theorem IV.2.14] Let f be analytic in $B(a;R) = \{z \in \mathbb{C} \mid |z-a| < R\}$ and suppose $|f(z)| \leq M$ for all $z \in B(a;R)$. Then $|f^{(n)}(a)| \leq n!M/R^n$.

Theorem. [Gentili and Struppa 2007, Theorem 3.6] Let $B = B(0;R) = \{q \in \mathbb{H} \mid |q| \leq R\}$ and suppose $f : B \to \mathbb{H}$ is a regular function. Let $r < R$, $I \in \mathbb{S}$, and $\partial\Delta_I(0,r) = \{(x+yI) \mid x^2+y^2 = r^2\}$. If $M_I = \max\{|f(q)| \mid q \in \partial\Delta_I(0,r)\}$ and if $M = \inf\{M_I \mid I \in \mathbb{S}\}$, then

$$\left| \left. \frac{\partial^n f}{\partial x^n} \right|_{q=0} \right| \leq \frac{n!M}{r^n}, \text{ for } n \geq 0.$$

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Liouville's Theorem

Theorem. [Conway Theorem IV.3.4] If f is a bounded entire function (i.e. analytic in all of $\mathbb{C}$), then f is constant.

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Structure of the Zero Set

Theorem. [Conway Corollary IV.3.10] If $f : G \to \mathbb{C}$ is analytic and not constant, $a \in G$, and $f(a) = 0$ then there is an $R > 0$ such that $B(a; R) \subset G$ and $f(z) \neq 0$ for $0 < |z - a| < R$.

Theorem. [Gentili and Struppa 2007, Theorem 5.3] If f has a series representation $f(q) = \sum_{n=0}^{\infty} q^n a_n$ with real coefficients a_n, then every real zero x_0 is isolated, and if $x_0 + y_0 I$ is a nonreal zero (i.e. $y_0 \neq 0$) then $x_0 + y_0 I$ is a zero for any $I \in \mathbb{S}$. In particular, if $f \not\equiv 0$, the zero set of f consists of isolated points (belonging to $\mathbb{R}$) or isolated 2-spheres.

Note. A similar result holds for the coefficients a_n as quaternions (but with some small additional hypotheses). See Gentili and Struppa's Theorem 5.4.

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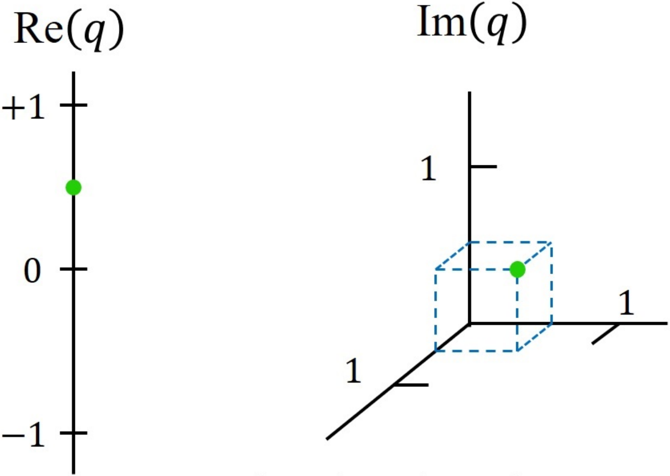
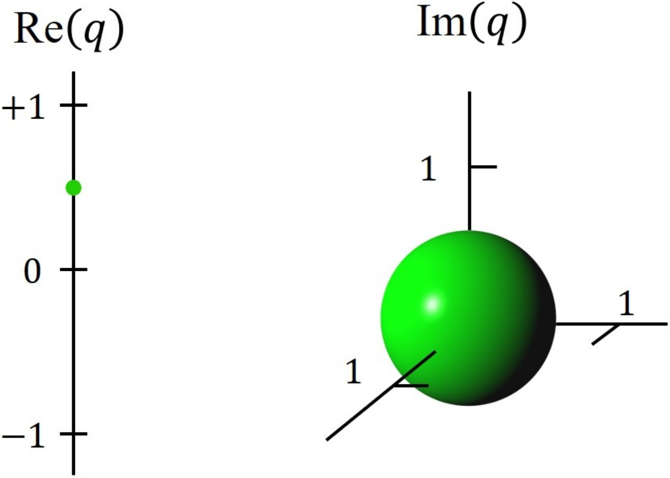


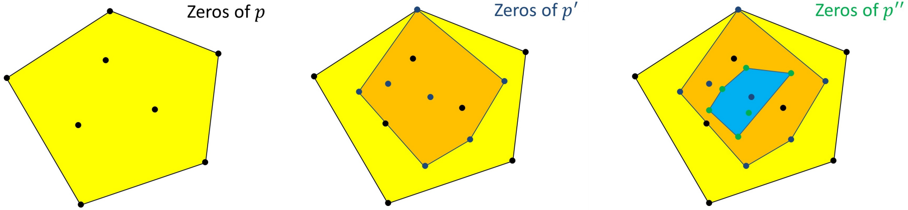
$$q = \frac{1}{2} + i\,\frac{1}{2} + j\,\frac{1}{2} + k\,\frac{1}{2}$$



$$q \in \frac{1}{2} + \frac{1}{2}\mathbb{S}$$

Complex Gauss-Lucas Theorem

Theorem. If all the zeros of a polynomial P lie in a half plane in the complex plane, then all zeros of the derivative P' lie in the same half plane.



Note. The Gauss-Lucas Theorem does not hold in general since the polynomial $p(q) = (q - i)(q - j) = q^2 - q(i + j) + k$ has zero set $\{i\}$, but $p'(q) = 2q - (i + j)$ has zero set $\{(i + j)/2\}$ [Ghiloni and Perotti, 2018].

Quaternionic Gauss-Lucas Theorem, Introduction

Definition. [Vlacci 2011, Definition 3.1] A domain $\Omega \subset \mathbb{H}$ is *axially symmetric* if for all $x + yI \in \Omega$, the whole 2-sphere $s + y\mathbb{S} \subset \Omega$. If $V \subset \mathbb{H}$ then the set $\tilde{V} = \cup_{x+yI \in V} x + y\mathbb{S}$ is the *axially symmetric completion* of V.

Note. The unit ball $B = \{q \in \mathbb{H} \mid |q| = 1\}$ is an axially symmetric set.

Definition. [Vlacci 2011, Definition 3.8] Let $f(q) = \sum_{n=0}^{\infty} q^n a_n$ be a given quaternionic power series with radius of convergence R. Define the *regular conjugate* of f as the series $f^c(q) = \sum_{n=0}^{\infty} q^n \overline{a}_n$. Define f^s as the regular product $f^s = ff^c = f^c f$; if f is a polynomial, this is sometimes called the *normal polynomial* of f [Ghiloni and Perotti, 2018].

Note. The coefficients of f^s are real. If f is a quaternionic polynomial of degree n, then f^s is a real polynomial of degree $2n$ [Vlacci, 2011].

Definition. For a set Z of elements of $\mathbb{H}$, the *convex hull* of Z is the intersection of all convex sets in $\mathbb{H}$ which contain Z, denoted $\mathcal{K}(Z)$.

Quaternionic Gauss-Lucas Theorem

Theorem. Gauss-Lucas in $\mathbb{H}$. [Vlacci 2011, Proposition 3.14] Let $p(q) = \sum_{r=0}^{n} q^r a_r$ be a polynomial with derivative $p'(q) = \sum_{r=1}^{n} q^{r-1} r a_r$. Let $p^s = pp^c$ be the normal polynomial of p and let Z_{p^s} be the zero set of p^s. Then all zeros of p' lie in the axially symmetric completion of the convex hull $\mathcal{K}(Z_{p^s})$.

Note. We saw above that the polynomial $p(q) = (q-i)(q-j) = q^2 - q(i+j) + k$ has zero set $\{i\}$, but $p'(q) = 2q - (i+j)$ has zero set $\{(i+j)/2\}$ and so violates a general Gauss-Lucas Theorem for quaternions. In this example, $p^c(q) = q^2 + q(i+j) + k$ and so

$$p^s(q) = p(q)p^c(q) = (q^2 - q(i+j) + k)(q^2 + q(i+j) - k) = (q^2+1)^2.$$

The zero set of p^s is then $\mathbb{S}$ and the convex hull $\mathcal{K}(Z_{p^s})$ is the "interior" of $\mathbb{S}$, $\mathcal{K}(Z_{p^s}) = \{q = 0 + ix_1 + jx_2 + kx_3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$ (which is already axially symmetric). Notice that $(i+j)/2 \in \mathcal{K}(Z_{p^2})$, as guaranteed by the Gauss-Lucas Theorem in $\mathbb{H}$.

Complex Bernstein Inequality

Definition/Theorem. For a complex polynomial P, a norm is given by $\|P\| = \max_{|z|=1} |P(z)|$.

Note. There are a number of other norms on complex polynomials. One collection of such norms are the L^p norms given by $\|P\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^P \, d\theta \right\}^{1/p}$.

Bernstein's Inequality. [Serge Bernstein, 1926] For a complex polynomial P of degree n, we have $\|P'\| \leq n\|P\|$.

Erdös-Lax Theorem. [Lax, 1944] If P is a complex polynomial of degree n such that $P(z) \neq 0$ for $|z| < 1$, then $\|P\| \leq n\|P\|/2$.

Note. For more details on these results, see *Bernstein-Type Inequalities for Polynomials*, R. Gardner, N. K. Govil, and G. Milovanovic, Elsevier Press (2019).

Quaternionic Bernstein Inequality

Theorem. [Gal and Sabadin 2015, Theorem 2.1] If P is a quaternionic polynomial of degree n, then $\|P'\| \leq n\|P\|$ where $\|P\| = \max_{|q|=1} |P(q)|$.

Note. The proof uses the Maximum Modulus Theorem and the Gauss-Lucas Theorem (the quaternionic versions, of course).

Quaternionic Erdös-Lax Theorem, Introduction

Theorem. [Gal and Sabadini 2015, Theorem 3.1] The Erdös-Lax inequality is not valid, in general, for quaternionic polynomials.

Note. The proof is based on the example mentioned above of $P(q) = (q - i)(q - j) = q^2 - q(i + j) + k$. We have claimed that the zero set for P is $\{i\}$. Gal and Sabadini give the computations to show that

$$\|P'\| \geq (6 + 2\sqrt{2})^{1/2} > (4 + 4\sqrt{2})^{1/2} \geq \|P\| = \frac{2}{2}\|P\| = \frac{n}{2}\|P\|.$$

Quaternionic Erdös-Lax Theorem

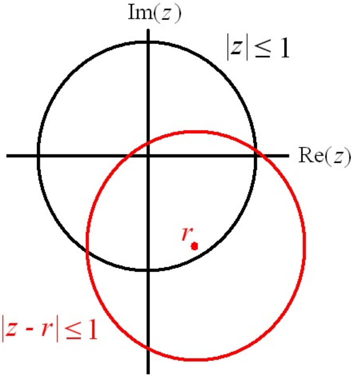
Theorem. [Gal and Sabadini 2015, Proposition 3.2] If P is a quaternionic polynomial of degree n that has no zero in the ball $|q| < 1$. Assume that the zeros of P are either spheres and/or real points and that there exists at most one isolated zero $\alpha \in \mathbb{H} \setminus \mathbb{R}$ that has multiplicity 1. Then $\|P'\| \leq n\|P\|/2$.

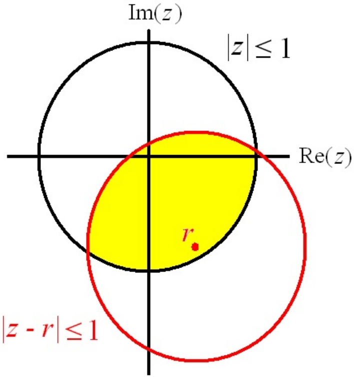
Note. The bound above is optimal, as seen by considering $P(q) = (1 + q^n)/2$.

Other Polynomial Complex Results 1

Centroid Theorem. The centroid of the zeros of a complex polynomial P is the same as the centroid of the zeros of P'.

Ilieff-Sendov Conjecture. If all the zeros of a polynomial P lie in $|z| \leq 1$ and if r is a zero of P, then there is a zero of P' in the circle $|z - r| \leq 1$.





Other Polynomial Complex Results 2

Rate of Growth Theorem. [Bernstein, 1926] If P is a complex polynomial of degree n such that $|P(z)| \leq M$ on $|z| = 1$, then for $R \geq 1$ we have $\max_{|z|=R} |P(z)| \leq MR^n$.

Note. The proof of the Rate of Growth Theorem only requires the Maximum Modulus Theorem (which holds in the quaternionic setting).

Eneström-Kakeya Theorem. [Eneström 1893, Kakeya 1912] If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a complex polynomial with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|z| < 1$.

Note. The proof of the Eneström-Kakeya Theorem only requires the Triangle Inequality for modulus and the Maximum Modulus Theorem (both of which hold in the quaternionic setting).

Other Polynomial Complex Results 2

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