Some Results on the Location of Zeros of a Polynomial (970-30-171)

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Abstract. The Eneström-Kakeya Theorem states that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial satisfying $0 < a_0 \leq a_1 \leq \ldots \leq a_n$, then all the zeros of $p(z)$ lie in $|z| \leq 1$. We present related results by considering polynomials with complex coefficients and by putting restrictions on the arguments and moduli of the coefficients.

HISTORY

Question. If $p(z)$ is a polynomial of degree n, then what are the zeros of $p(z)$?

Answer 1. If $n = 1, 2, 3,$ or 4, then no problem (antiquity).

Answer 2. If $n \geq 5$, then **problem!** (Abel *et al.*)

Question. If $p(z) = \sum_{v=0}^{n} a_v z^v$, then what restrictions can be put on the location of the zeros of $p(z)$ in the complex plane?

Theorem 1. (Cauchy 1830)
All the zeros of $p(z) = \sum_{v=0}^{n} a_v z^v$, where $a_n \neq 0$, lie in the circle $|z| < 1+M$, where $M = \max_{0 \leq j \leq (n-1)} \left|\frac{a_j}{a_n}\right|$.

Theorem 2. (Eneström-Kakeya 1920)
If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with real coefficients satisfying

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_n,$$

then all the zeros of $p(z)$ lie in $|z| \leq 1$.

Theorem 3. (Joyal-Labelle-Rahman 1967)
If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with real coefficients, $a_n \neq 0$, satisfying

$$a_0 \leq a_1 \leq \cdots \leq a_n,$$

then all the zeros of $p(z)$ lie in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Note. If $a_0 > 0$, then Theorem 3 reduces to Theorem 2.

Theorem 4. (Govil and Rahman 1968)
If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial such that

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \text{ for } j \in \{0, 1, \ldots, n\} \text{ for some real } \beta,$$

and

$$|a_0| \leq |a_1| \leq \cdots \leq |a_n|,$$

then all the zeros of $p(z)$ lie in $|z| \leq R$ where

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{v=0}^{n-1} |a_v|.$$

Note. The following result for *analytic functions* has a rather flexible condition on the coefficients of the series expansion of the function.

Theorem 5. (Aziz and Mohammed 1980)
Let $f(z) = \sum_{v=0}^{\infty} a_v z^v$ be analytic in $|z| \leq t$. If $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$ for $j = 0, 1, \ldots$, for some k and r, and for some $t \geq 0$,

$$0 < \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k \alpha_k \geq t^{k+1} \alpha_{k+1} \geq \cdots$$

and

$$\beta_0 \leq t\beta_1 \leq \cdots \leq t^r \beta_r \geq t^{r+1} \beta_{r+1} \geq \cdots$$

then $f(z) \neq 0$ in

$$|z| < \frac{t|a_0|}{2(\alpha_k t^k + \beta_r t^r) - (\alpha_0 + \beta_0)}.$$

Note. We put a similar condition on the coefficients of a *polynomial.*

FIRST RESULT

Theorem A. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial such that

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \text{ for } j \in \{0, 1, \ldots, n\} \text{ for some real } \beta,$$

and for some positive t and some nonnegative integer K,

$$|a_0| \leq t|a_1| \leq \cdots \leq t^K|a_K| \geq t^{K+1}|a_{K+1}| \geq \cdots \geq t^n|a_n|,$$

then all the zeros of $p(z)$ lie in $|z| \geq R$ where

$$R = \min\left\{\frac{|a_0|t}{(2|a_K|t^K - |a_0|)\cos\alpha + |a_0|\sin\alpha + 2\sin\alpha\sum_{v=1}^{n-1}|a_v|t^v + t^n|a_n|(1+\sin\alpha - \cos\alpha)}, t\right\}.$$

Note. By applying Theorem A to $q(z) = z^n p\left(\frac{1}{z}\right)$, we get the following generalization of Theorem 4 (Govil and Rahman):

Theorem B. If $q(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial such that

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \text{ for } j \in \{0, 1, \ldots, n\} \text{ for some real } \beta,$$

and if for some positive t and some nonnegative integer K,

$$|a_n| \leq t|a_{n-1}| \leq \cdots \leq t^K|a_{n-K}| \geq t^{K+1}|a_{n-K-1}| \geq \cdots \geq t^n|a_0|,$$

then all the zeros of $q(z)$ lie in $|z| \leq R$ where

$$R = \max\left\{\frac{(2|a_{n-K}|t^K - |a_n|)\cos\alpha + |a_n|\sin\alpha + 2\sin\alpha\sum_{v=1}^{n-1}|a_{n-v}|t^v + t^n|a_0|(1+\sin\alpha - \cos\alpha)}{|a_n|t}, \frac{1}{t}\right\}.$$

Note. Notice that with $t = 1$ and $K = 0$ in Theorem B, we get:

Corollary C. If $q(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial such that

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \text{ for } j \in \{0, 1, \ldots, n\} \text{ for some real } \beta,$$

and

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_0|,$$

then all the zeros of $q(z)$ lie in $|z| \leq R$ where

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{v=1}^{n-1} |a_{n-v}| + \frac{|a_0|}{|a_n|}(1 + \sin\alpha - \cos\alpha).$$

Note. Notice that, since $1 + \sin\alpha - \cos\alpha \leq 2\sin\alpha$ when $0 < \alpha \leq \pi/2$, Corollary C is an improvement of Theorem 4 (Govil and Rahman) (and when $\alpha = 0$, both of these results reduce to Theorem 2 of Eneström-Kakeya), though Corollary C is inherent in the proof of Theorem 4 (Govil and Rahman).

Note. We can extract further corollaries from Theorems A and B by choosing $K \in \{0, n\}$ and $t \in \{t, 1\}$.

SECOND RESULT

Note. The proof of Theorem A will employ Schwarz's Lemma. We can use a generalization of Schwarz's Lemma to produce a result which, although not as concise as Theorem A, can produce bounds on the zero containing region of a polynomial which are better than those of Theorem A.

Theorem D If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial such that

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \text{ for } j \in \{0, 1, \ldots, n\} \text{ for some real } \beta,$$

and for some positive t and some nonnegative integer K,

$$|a_0| \leq t|a_1| \leq \cdots \leq t^K |a_K| \geq t^{K+1}|a_{K+1}| \geq \cdots \geq t^n |a_n|,$$

then all the zeros of $p(z)$ lie in $|z| \geq R$ where

$$\begin{aligned} R &= \min\left\{\frac{-|b|t^2(M - |a_0|t) + \{t^4|b|^2(M - |a_0|t)^2 + 4|a_0|M^3t^3\}^{1/2}}{2M^2}, t\right\} \\ M &= t|a_0|\left\{\left(2\left|\frac{a_K}{a_0}\right|t^K - 1\right)\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{v=0}^{n-1}|a_v|t^v \right. \\ &\quad \left. +t^n\left|\frac{a_n}{a_0}\right|(1 + \sin\alpha - \cos\alpha)\right\} \\ b &= a_0 - ta_1. \end{aligned}$$

AN EXAMPLE.

Note. The proof of Theorem D is similar to the proof of Theorem A, but uses a generalization of Schwarz's Lemma whereas the proof of Theorem A uses Schwarz's Lemma. One is therefore lead to believe that Theorem D should give better results than Theorem A. Due to the complicated nature of the parameters in these theorems, it's difficult to compare the results directly. However, we can give an example to show that Theorem D *can* give better bounds than does Theorem A.

Example. Consider the polynomial $p(z) = \frac{1}{10} + 2z + 4z^2 + 8z^3$. By Theorem A with $t = \frac{1}{2}$ and $K = 1$, we get that $p(z) \neq 0$ for $|z| < \frac{1}{38}$. By Theorem D with $t = \frac{1}{2}$ and $K = 1$, we get that $p(z) \neq 0$ for $|z| < .04826$. This is an improvement of Theorem A by a factor of about 3.36 (in terms of area).

PROOF OF THEOREM A.

Proof. Without loss of generality, we may assume $\beta = 0$. Consider

$$\begin{aligned} P_1(z) = (z-t)p(z) &= -ta_0 + z\sum_{v=1}^{n}(a_{v-1} - ta_v)z^{v-1} + a_n z^{n+1} \\ &\equiv -ta_0 + G_1(z). \end{aligned}$$

Now since $|\arg a_j| \leq \alpha \leq \pi/2$ for $j \in \{0, 1, \ldots, n\}$, then by a lemma of Aziz and Mohammad, for $|z| = t$

$$|ta_j - a_{j-1}| \leq |t|a_j| - |a_{j-1}||\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha.$$

Therefore for $|z| = t$,

$$\begin{aligned} |G_1(z)| &\leq t\sum_{v=1}^{n}|a_{v-1} - ta_v|t^{v-1} + t^{n+1}|a_n| \\ &\leq t\left\{\sum_{v=1}^{n}|t|a_v| - |a_{v-1}||t^{v-1}\cos\alpha + \sum_{v=1}^{n}(t|a_v| + |a_{v-1}|)t^{v-1}\sin\alpha + t^n|a_n|\right\} \\ &= t|a_0|\left\{\left(2\left|\frac{a_K}{a_0}\right|t^K - 1\right)\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{v=1}^{n-1}|a_v|t^v \right. \\ &\qquad \left. + t^n\left|\frac{a_n}{a_0}\right|(1 + \sin\alpha - \cos\alpha)\right\} \\ &\equiv t|a_0|M \end{aligned} \tag{1}$$

Now since $G_1(0) = 0$, then it follows from Schwarz's Lemma that

$$|G_1(z)| \leq |a_0|M|z| \text{ for } |z| \leq t.$$

So by (1)

$$\begin{aligned} |P_1(z)| &\geq t|a_0| - |G_1(z)| \\ &\geq |a_0|[t - |z|M] \text{ for } |z| \leq t. \end{aligned}$$

Therefore $P_1(z) > 0$ if $|z| < \dfrac{t}{M}$ and $|z| < t$. The result follows. ∎

LEMMAS

Lemma 1. (Aziz and Mohammad)
Let $f(z) = \sum\limits_{v=0}^{\infty} a_v z^v$ be analytic in $|z| \leq t$ such that

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \text{ for } j \in \{0, 1, \ldots\} \text{ for some real } \beta,$$

and if for positive t and some nonnegative integer K,

$$|a_0| \leq t|a_1| \leq \cdots \leq t^K|a_K| \geq t^{K+1}|a_{K+1}| \geq \cdots,$$

then for $j \in \{1, 2, \ldots\}$

$$|ta_j - a_{j-1}| \leq |t|a_j| - |a_{j-1}|| \cos\alpha + (t|a_j| + |a_{j-1}|) \sin\alpha.$$

Lemma 2. (Schwarz's Lemma)
If $f(z)$ is analytic for $|z| < r$ and satisfies the conditions $|f(z)| \leq M$ and $f(0) = 0$, then $f(z)| \leq M|z|/r$ for $|z| < r$.

Lemma 3. Generalized Schwarz's Lemma (Govil, Rahman, Scheisser)
If $f(z)$ is analytic in $|z| \leq r$, $f(0) = f'(0) = b$, and $|f(z)| \leq M$ for $|z| = r$, then for $|z| \leq r$

$$|f(z)| \leq \frac{M|z|}{r^2} \frac{M|z| + r^2|b|}{M + |z||b|}.$$