# Introductory Level Analysis: Synthesizing $\mathbb{R}$, $\mathbb{R}^n$, Metric Spaces and Topological Spaces

Bob Gardner
Department of Mathematical and Statistical Sciences
(DoMaSS)
East Tennessee State University

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# ABSTRACT

All too often, undergraduate math majors get a thorough exposure to the structure of $\mathbb{R}$ (or maybe $\mathbb{R}^n$), but are left entirely unexposed to the same concepts (eg. "open," "closed," "compact," "connected," "limit," and "continuity") in more general settings, such as metric spaces and topological spaces. This is unfortunate, since in most instances, it is possible to extend many of the results from $\mathbb{R}$ to $\mathbb{R}^n$, metric spaces and topological spaces, with little to no revision (provided the relevant definitions are introduced). The purpose of this presentation is to show how a standard introductory real analysis class can be used, with minimal interruption, to extend many results from $\mathbb{R}$ to these more general settings.

# 1. A STANDARD INTRODUCTION TO REAL ANALYSIS

**Note.** Depending on other classes available in the curriculum, the standard introductory real analysis class (hence forth called "senior analysis") covers the following topics:

**PART I.**

- Introduction to Proof Techniques
- Naive Set Theory and Functions
- Axiomatic Set Theory, Functions, and Cardinality
- Axiomatic Development of the Real Numbers (algebraic properties, completeness, rationals, irrationals, algebraic numbers)
- Topology of $\mathbb{R}$ (open/closed sets, limit points, boundary points, compactness, Heine-Borel Theorem, connectedness)
- Sequences and Convergence (limits, divergence, monotonicity, boundedness, subsequences, subsequential limits, the Bolzano-Weierstrass Theorem)
- Functions (limits, continuity, inverse images, Extreme Value Theorem, Intermediate Value Theorem, Uniform Continuity)

**PART II.**

- Differentiation (Mean Value Theorems)
- Riemann Integration (properties, necessary and sufficient conditions for Riemann integrability, Riemann-Stieltjes integration)
- Sequences and Series of Functions (convergence, Taylor Series, Fourier Series [maybe])

In this presentation, we explore ways to extend the topics of Part I to a more general setting than $\mathbb{R}$. Suggestions are given as to how to present the new material “in parallel” with the standard material. Specific examples are presented to show some surprising applications of the new material.

## 2. $\mathbb{R}$ AND SOME OF ITS COLLEAGUES

**Definition.** For this presentation, we define $\mathbb{R}$ as a complete, ordered field.

**Fields.** By their senior year, students should be familiar with the following fields (at least intuitively):

- $\mathbb{R}$, the real numbers
- $\mathbb{Q}$, the rational numbers
- $\mathbb{C}$, the complex numbers
- $\mathbb{Z}_p$ where $p$ is prime, the integers modulo $p$

$\mathbb{Q}$ is an interesting example, since it is ordered but not complete. $\mathbb{C}$ is complete, but not ordered (which will later raise the question "What does it mean for a field to be complete if it is not ordered?"). $\mathbb{Z}_p$, $p$ prime, is a finite field and not in the "realm of analysis."

**Orderings.** Recall that the "Order Axiom" postulates the existence of a positive set $P$ such that

- If $a, b \in P$ then $a + b \in P$ and $a \cdot b \in P$ (Closure under $+$ and $\cdot$).
- For any $a$ in the field, exactly one of the following holds: $a \in P$, $-a \in P$, or $a = 0$ (Law of Trichotomy).

Certainly $\mathbb{R}$ and $\mathbb{Q}$ (intuitively) are ordered. However, $\mathbb{C}$ and $\mathbb{Z}_p$ are not ordered. We might also observe that there is no clear ordering on $\mathbb{R}^n$.

**Completeness.** Recall that the Axiom of Completeness states that any set of real numbers with an *upper* bound has a *least upper* bound. However, this makes explicit use of the ordering. Intuitively, completeness of a space implies that there are "no holes" in the space. We see from the definition that $\mathbb{Q}$ is not complete. However, we claim that $\mathbb{C}$ is complete, though it has no ordering. We can also claim that $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete, and they are not ordered. These observations imply that there may be other approaches to completeness which do not involve orderings. However, we must explore sequences before this idea can be further studied.

# 3. TOPOLOGY

**Recall.** A set $A \subset \mathbb{R}$ is *open* if for all $x \in A$, there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset A$. This can be easily extended to a metric space $(M, \rho)$: A set $A \subset M$ in a metric space is *open* if for all $x \in A$, there exists $\epsilon > 0$ such that $B(x, \epsilon) = \{y \mid \rho(x, y) < \epsilon\} \subset A$. This allows us to discuss open sets without appealing to an ordering. It also shows how the idea of "open" in $\mathbb{R}$ can be extended to other settings, such as $\mathbb{R}^n$ (familiar from sophomore level Linear Algebra) and $\mathbb{C}$.

**Note.** Of course, in senior analysis we prove:

**(1)** $\emptyset$ and $\mathbb{R}$ are open.

**(2)** If $A_1, A_2, \dots, A_N$ are open, then $\bigcap_{i=1}^{N} A_i$ is open.

**(3)** If $\{A_i\}_{i \in I}$ is any collection of open sets, then $\bigcup_{i \in I} A_i$ is open.

With this result as inspiration (I call it the "logical ancestry"), we can define a topological space:

A *topological space* $(\mathcal{T}, \mathcal{O})$ is a point set $\mathcal{T}$ and a set $\mathcal{O} \subset \mathcal{P}(\mathcal{T})$ of *open* sets such that:

**(1)** $\emptyset$ and $\mathcal{T}$ are open (i.e. $\emptyset, \mathcal{T} \in \mathcal{O}$.)

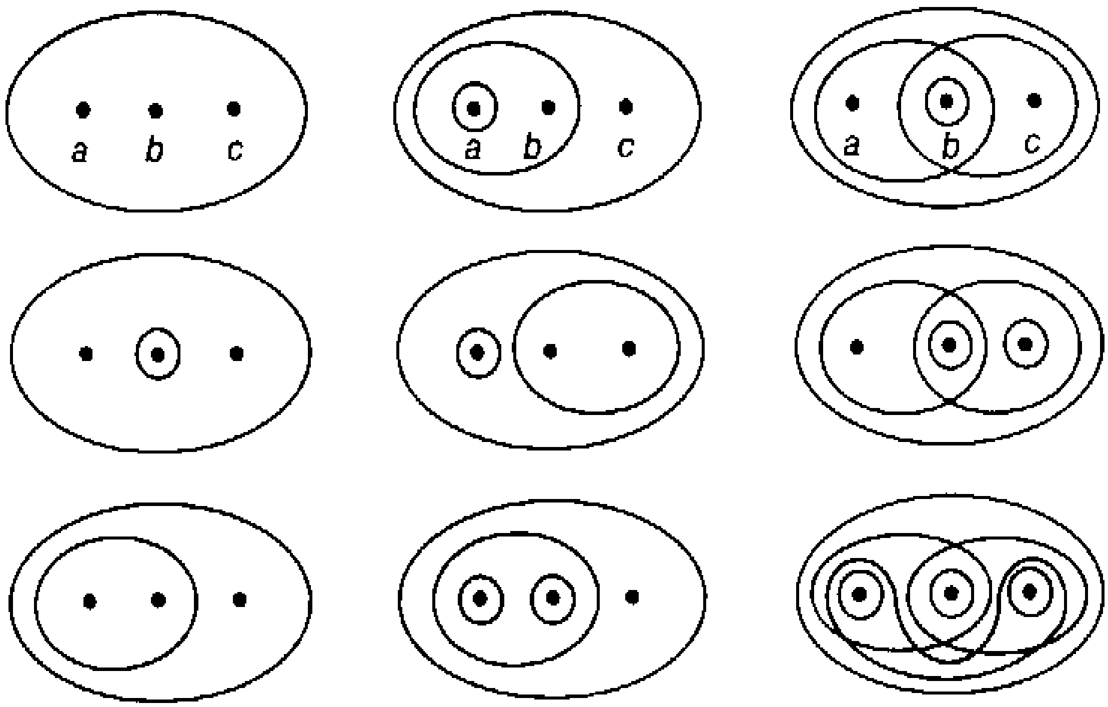
**(2)** If $\{O_1, O_2, \dots, O_N\} \subset \mathcal{O}$, then $\bigcap_{i=1}^{N} O_i \in \mathcal{O}$.

**(3)** If $\{O_i\}_{i \in I} \subset \mathcal{O}$ then $\bigcup_{i \in I} O_i \in \mathcal{O}$.

We can easily introduce a few examples of topological spaces:

- $\mathbb{R}$ under the *usual topology* (the one studied in senior analysis).
- $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, the *discrete topology* on $\mathbb{R}$.
- $(\mathbb{R}, \{\emptyset, \mathbb{R}\})$, the *trivial topology* on $\mathbb{R}$.

For the sake of illustration, we can mention some topologies on a set of cardinality 3: $\{a, b, c\}$




From *Topology, A First Course* by J. R. Munkres.

**Note.** We can draw parallels between definitions in $\mathbb{R}$ which use the ordering (in the sense that these definitions involve intervals $(a, b) = \{x \mid a < x < b\}$), definitions in a metric space $(M, \rho)$, and definitions in a topological space $(\mathcal{T}, \mathcal{O})$:

| **TERM** | $\mathbb{R}$ | **Metric Space** | **Topological Space** |
|---|---|---|---|
| closed set | Set $X$ is *closed* if $X^c$ is open. | Set $X$ is *closed* if $X^c$ is open. | Set $X$ is *closed* if $X^c$ is open. |
| limit point | Point $x$ is a *limit point* of set $X$ if for all $\epsilon > 0$, the interval $(x - \epsilon, x + \epsilon)$ contains infinitely many points of $X$. | Point $x$ is a *limit point* of set $X$ if for all $\epsilon > 0$, the set $B(x, \epsilon)$ contains infinitely many points of $X$. | Point $x$ is a *limit point* of set $X$ if all open sets $O_x$ containing $x$ also contains infinitely many points of $X$. |
| boundary point | Point $x$ is a *boundary point* of set $X$ if for all $\epsilon > 0$, the interval $(x - \epsilon, x + \epsilon)$ contains a point in $X$ and a point in $X^c$. | Point $x$ is a *boundary point* of set $X$ if for all $\epsilon > 0$, the set $B(x, \epsilon)$ contains a point in $X$ and a point in $X^c$. | Point $x$ is a *boundary point* of set $X$ if all open sets $O_x$ containing $x$ contain a point in $X$ and a point in $X^c$. |
| connected set | A *separation* of set $X$ is two open sets $U$ and $V$ such that (1) $U \cap V = \emptyset$ (2) $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$ (3) $(U \cap A) \cup (V \cap A) = A$. Set $X$ is *connected* if there is not separation of $X$. | A *separation* of set $X$ is two open sets $U$ and $V$ such that (1) $U \cap V = \emptyset$ (2) $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$ (3) $(U \cap A) \cup (V \cap A) = A$. Set $X$ is *connected* if there is not separation of $X$. | A *separation* of set $X$ is two open sets $U$ and $V$ such that (1) $U \cap V = \emptyset$ (2) $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$ (3) $(U \cap A) \cup (V \cap A) = A$. Set $X$ is *connected* if there is not separation of $X$. |
| compact set | Set $X$ is *compact* if every open cover of $X$ has a finite subcover. | Set $X$ is *compact* if every open cover of $X$ has a finite subcover. | Set $X$ is *compact* if every open cover of $X$ has a finite subcover. |

Many of the topological results about $\mathbb{R}$ (such as "closed sets contain their limit points") can then easily be extended to more general settings.

**Note.** Not all topological results of $\mathbb{R}$ are so easily extended:

- An open set of real numbers is a countable disjoint union of open intervals. This result, useful when first encountering measure theory, has no easy parallel in, say, $\mathbb{R}^n$. However, it can be used to motivate a discussion of the *basis* of a topology.
- Connected sets in $\mathbb{R}$ are intervals. This result very much depends on the "one-dimensional nature" of $\mathbb{R}$ — when considering $\mathbb{C}$ or $\mathbb{R}^n$ $(n \geq 2)$, we must explore simple and multiple connectedness.
- The Heine-Borel Theorem: A set of real numbers is compact if and only if it is closed and bounded. Though this result is true in many familiar settings ($\mathbb{R}$ and $\mathbb{R}^n$), there are metric spaces where the result does not hold.

## 4. $l^2$ — A NICE SPACE FOR COUNTEREXAMPLES

**Definition.** The space $l^2$ consists of all square summable sequences:

$$l^2 = \{(x_1, x_2, x_2, \dots) \mid x_1, x_2, x_3, \dots \in \mathbb{R}, x_1^2 + x_2^2 + \dots < \infty\}.$$

We claim that $l^2$ is an infinite dimensional vector space with *norm*

$$\|(x_1, x_2, x_3, \dots)\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}$$

and *metric*

$$d(\vec{x}, \vec{y}) = \|(x_1, x_2, x_3, \dots) - (y_1, y_2, y_2, \dots)\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}.$$

**Note.** Space $l^2$ gives us a setting to show that not everything is as straightforward as it is in $\mathbb{R}$:
The Heine-Borel Theorem does not hold in $l^2$. There exists a set $X$ which is closed and bounded, but is not compact: consider

$$X = \{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, 0, \dots), \cdots\}$$

and open covering

$$\{ \text{ open balls with center } x \text{ and radius } 1/2 \mid x \in X\}.$$

Then set $X$ is open (consider $X^c$), but the open cover has no finite subcover (since each $x \in X$ is contained in exactly one of the open sets).

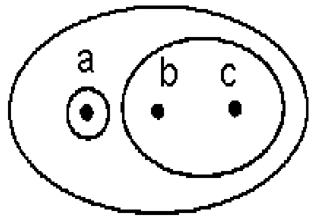
# 5. SEQUENCES AND CONVERGENCE

**Definition.** Let $x_1, x_2, x_3, \ldots$ be a sequence. We define the *limit* of the sequence and define the sequence to be *Cauchy* if

| Definition in $\mathbb{R}$ | in a metric space | in a topological space |
|---|---|---|
| Sequence $(x_n)$ *converges* to $L$ if for all $\epsilon > 0$, there exists natural number $N$ such that if $n$ (a natural number) is greater than $N$, then $x_n \in (L - \epsilon, L + \epsilon)$. | Sequence $(x_n)$ *converges* to $L$ if for all $\epsilon > 0$, there exists natural number $N$ such that if $n$ (a natural number) is greater than $N$, then $x_n \in B(L, \epsilon)$. | Sequence $(x_n)$ *converges* to $L$ if for all open sets $O_x$ containing $L$, there exists natural number $N$ such that if $n$ (a natural number) is greater than $N$, then $x_n \in O_x$. |
| Sequence $(x_n)$ is *Cauchy* if for all $\epsilon > 0$, there exists a natural number $N$ such that for all natural numbers $n, m > N$ we have $\lvert x_n - x_m \rvert < \epsilon$. | Sequence $(x_n)$ is *Cauchy* if for all $\epsilon > 0$, there exists a natural number $N$ such that for all natural numbers $n, m > N$ we have $\rho(x_n, x_m) < \epsilon$. | — |

(It is rather surprising that we can talk about limits [and hence continuity] in a topological space which may not have a metric!)

**Note.** In senior analysis, we prove such "obvious" results as "the limit of a sequence of real numbers is unique." Surprisingly, this is not the case in all settings. Consider the topological space:




From *Topology, A First Course* by J. R. Munkres.

The sequences $b, b, b, \ldots$ and $c, c, c, \ldots$ each converge to both $b$ and to $c$. (This can be used to motivate a discussion of Hausdorff spaces.) Also, under the discrete topology, the only convergent sequences are those which are eventually constant.

**Note.** The Bolzano-Weierstrass Theorem states that an infinite bounded set of real numbers has a limit point. This is true in $\mathbb{R}^n$ and $\mathbb{C}$. However, consider the set $X$ in $l^2$:

$$X = \{(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, 0, \ldots), \cdots\}.$$

The set is infinite and bounded (each point is within 1 unit of the "origin"), but there is not a limit point (in fact, each point is an isolated point within $\sqrt{2}$ units of each of the other points). From this example, we see that there is a lot of "room" in the infinite dimensional space $l^2$ — we are able to choose an infinite number of points and still not have them "cluster"... this is done by choosing the points in different "directions" from the origin.

# 6. CAUCHY AND COMPLETENESS

**Note.** We have seen that completeness in $\mathbb{R}$ is defined in terms of upper bounds and least upper bounds. However, this definition cannot be extended to a setting where there is no ordering.

**Note.** A sequence of real numbers is *Cauchy* if and only if it is convergent. In fact, we can show that in a metric space, every convergent sequence is Cauchy. However, there exist metric spaces in which Cauchy sequences may not converge (such as $\mathbb{Q}^n$ with the usual metric).

**Definition.** A metric space is *complete* if every Cauchy sequence converges.

**Note.** We can show that $\mathbb{R}^n$ (or $\mathbb{C}^n$) is complete using the fact that $\mathbb{R}$ (or $\mathbb{C}$) is complete. However, the completeness of other spaces (such as $l^2$) can be a bit more difficult to prove directly.

# 7. FUNCTIONS

**Recall.** A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at point $x$ if:

for all $\epsilon > 0$, there exists $\delta > 0$ such that
if $x_0$ is in the domain of $f$ and $x_0 \in (x - \delta, x + \delta)$ then
$f(x_0) \in (f(x) - \epsilon, f(x) + \epsilon)$.

**Definition.** If $f : X \to Y$ where $(X, \rho_x)$ and $(Y, \rho_y)$ are metric spaces, then we can mimic the above definition: Such a function $f$ is continuous if

for all $\epsilon > 0$, there exists $\delta > 0$ such that
if $x_0$ is in the domain of $f$ and $\rho_x(x, x_0) < \delta$ then
$\rho_y(f(x), f(x_0)) < \epsilon$.

**Note.** In senior analysis, we see that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for each set $Y$ open (relative to the range of $f$), $f^{-1}(Y)$ is open (relative to the domain of $f$). This property is the motivation for the definition of continuity in the topological space setting: Let $f : \mathcal{T}_1 \to \mathcal{T}_2$ where $(\mathcal{T}_1, \mathcal{O}_1)$ and $(\mathcal{T}_2, \mathcal{O}_2)$ are topological spaces. Then $f$ is *continuous* if for each $O_2 \in \mathcal{O}_2$, we have $f^{-1}(O_2) \in \mathcal{O}_1$. This definition can then be used to prove an $\epsilon/\delta$ type result for such continuous functions: $f : \mathcal{T}_1 \to \mathcal{T}_2$ is continuous at $x \in \mathcal{T}_1$ if and only if for all open $O_\epsilon \in \mathcal{O}_2$ with $f(x) \in O_\epsilon$, there exists an open $O_\delta \in \mathcal{O}_1$ with $x \in O_\delta$ such that if $x_0 \in O_\delta$ then $f(x_0) \in O_\epsilon$.

**Note.** Since the definitions of many of the ideas are similar in $\mathbb{R}$, metric spaces, and topological spaces, there are easy extensions of familiar results of continuous functions from the setting of $\mathbb{R}$ to these other settings. For example, the following results hold in $\mathbb{R}^n$, metric spaces, and topological spaces:

- If $X$ is a compact set and $f$ is continuous, then $f(X)$ is compact.
- If $X$ is a connected set and $f$ is continuous, then $f(X)$ is connected.

## 8. CONCLUSION

**Note.** Historically, the ideas mentioned in this presentation were first explored in the real numbers. These ideas were generalized to the setting of metric spaces, and later to topological spaces. Due to this "genealogical relationship," we can often extend ideas studied in senior level analysis to these more general settings in a rather natural way. Students can then see the ideas in a broader setting, and see some foreshadowing of topics to be studied in more advanced classes.