Dealing with Infinite Dimensional Vector Spaces in Sophomore Level Linear Algebra

Bob Gardner
Department of Mathematics
Department of Physics and Astronomy
Institute of Mathematical and Physical Sciences
East Tennessee State University

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INTRODUCTION

Note. A great deal of applied mathematics depends on the concept of a **vector space**. In particular, many applications in ODEs and PDEs involve infinite dimensional vector spaces.

Note. In sophomore linear algebra, it is traditional to deal at length with finite dimensional vector spaces. However, infinite dimensional vector spaces are mentioned (at most) in passing. The purpose of this presentation is to give a method for addressing this oversight and for presenting infinite dimensional vector spaces (in particular l^2) as a natural follow-up to the finite dimensional cases. By making analogies between l^2 and $\mathbb{R}^n$, we get some **geometric** insight into function spaces (such as the space generated by the Fourier functions).

FINITE DIMENSIONAL VECTOR SPACES

Note. The approaches to vector spaces in the numerous introductory texts is surprisingly varied. Some texts introduce vectors first (and occasionally are unclear on the distinction between vectors in $\mathbb{R}^n$ and points in $\mathbb{R}^n$), some introduce matrices and matrix operations first, and some introduce systems of equations first. All eventually formally define "vector space." The better texts [my favorite one is Fraleigh and Beauregard's *Linear Algebra*] define the *coordinitization o*f vectors with respect to a given basis, *change of bases*, and *vector space isomorphism*. Most of these texts (some much more successfully than others) state and prove that an n-dimensional vector space over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n$.

Note. This result is quite amazing, really. We can classify a finite dimensional vector space simply by knowing its dimension and its scalar field! I therefore propose raising this theorem to the status of a "fundamental theorem":

THEOREM. Fundamental Theorem of Finite Dimensional Vector Spaces

An n-dimensional vector space with scalar field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.

Note. Since students almost exclusively deal with $\mathbb{R}^n$ in their sophomore level classes, they come to expect that every vector space has a **norm** and **inner product** on it. Of course, since "things are kept finite" in $\mathbb{R}^n$, divergence is not an issue (and it is the avoidance of these analysis ideas of convergence and divergence that make it possible to address finite dimensional vector spaces in a sophomore level class).

MODIFIED DEFINITIONS IN THE INFINITE DIMENSIONAL SETTING

Note. If we insist on defining a linear combination of vectors as a **finite** sum

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k,$$

then we find that the concept of an infinite dimensional vector space is quite complicated. In fact, the proof that every vector space has a basis (called a *Hamel basis* if we restrict ourselves to finite sums) requires the Axiom of Choice (Zorn's Lemma, actually). This **really means** that it may be impossible to **construct** a Hamel basis for a vector space, and therefore the application of vector spaces is severely limited with these restrictions! Hence, we modify several definitions and are, as a consequence, lead to a concept which is much more useful.

Definition. In an infinite dimensional vector space, a *linear combination* is the **infinite** sum $\sum_{k=1}^{\infty} s_k \mathbf{v}_k$. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots\}$ is *linearly independent* if

$$\sum_{k=1}^{\infty} s_k \mathbf{v}_k = \mathbf{0} \text{ implies } s_k = 0 \text{ for all } k.$$

The *span* of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots\}$ is the set of all (infinite) linear combinations of the vectors:

$$\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \dots\}) = \{s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots \mid s_k \in \mathbb{R} \text{ for all } k\}.$$

A *basis* of an infinite dimensional vector space is a linearly independent spanning set.

Note. The above definition of *basis* is sometimes called a *Schauder basis*. With this definition, we can naturally extend the Fundamental Theorem of Vector Spaces.

l^2 — A NATURAL PLACE TO LIVE!

Note. With the observation that an n-dimensional vector space is isomorphic to $\mathbb{R}^n$, it is natural to guess that an infinite dimensional vector space should be isomorphic to

$$\mathbb{R}^\infty = \{(r_1, r_2, \dots) \mid r_k \in \mathbb{R} \text{ for all } k\}.$$

However, there's clearly some problems if we want to preserve our familiar ideas of norms and inner products. For example, how long is the vector $(1, 1, 1, \dots)$?

Note. We can consider the subspace l^2 of $\mathbb{R}^\infty$:

$$l^2 = \left\{ (r_1, r_2, \dots) \mid r_k \in \mathbb{R} \text{ for all } k \text{ and } \sum_{k=1}^{\infty} r_k^2 < \infty \right\}.$$

In order to show that l^2 **is** actually a subspace, we only need to show that for all $\mathbf{v}_1, \mathbf{v}_2 \in l^2$:

1. $r\mathbf{v}_1 \in l^2$ for any $r \in \mathbb{R}$ (trivial), and

2. $\mathbf{v}_1 + \mathbf{v}_2 \in l^2$.

For the second result, it is sufficient to show that if $\sum_{k=1}^{\infty} r_k^2 < \infty$ and $\sum_{k=1}^{\infty}(r'_k)^2 < \infty$ then $\sum_{k=1}^{\infty}(r_k + r'_k)^2 < \infty$. This can be shown using mathematical induction and limits. Hence, it is possible to show that l^2 **is** a vector space. Only slightly more challenging is the proof of the following which completes our Fundamental Theorem:

Theorem. "Riesz/Fisher Theorem" or "The Fundamental Theorem of Infinite Dimensional Vector Spaces"
A complete infinite dimensional vector space with an inner product and with basis $\{\mathbf{b}_1, \mathbf{b}_2, \dots\}$ is isomorphic to l^2.

Note. We need to address three ideas before we can prove this result:

1. *Completeness* (usually touched on in the appendices of calculus books),

2. An *inner product* on a vector space (with which students develop familiarity in finite dimensions), and

3. *Isomorphism* between inner product spaces.

With these ideas established, the proof of the Fundamental Theorem is accessible!

Lemma 1. Pythagorean Formula. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are orthogonal vectors in an inner product space, then

$$\left\| \sum_{k=1}^{n} \mathbf{v}_k \right\|^2 = \sum_{k=1}^{n} \|\mathbf{v}_k\|^2.$$

Proof. Follows by induction from the properties of an inner product space. ∎

Definition. An inner product space H is *complete* if every Cauchy sequence converges. That is, for every sequence $\{\mathbf{v}_n\}$ such that

for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
if $m, n > N$ then $\|\mathbf{v}_n - \mathbf{v}_m\| < \epsilon$,

there exists $\mathbf{v} \in H$ such that $\mathbf{v}_n \to \mathbf{v}$.

Lemma 2. Let $\{\mathbf{v}_k\}$ be an orthonormal sequence in a complete inner product space H (i.e. H is a *Hilbert space*) and let $\{s_k\}$ be a sequence of real numbers. Then the series $\sum_{k=1}^{\infty} s_k\mathbf{v}_k$ converges if and only if $\sum_{k=1}^{\infty}(s_k)^2 < \infty$, and in that case

$$\left\| \sum_{k=1}^{\infty} s_k\mathbf{v}_k \right\|^2 = \sum_{k=1}^{\infty} |s_k|^2.$$

Proof. For every $m > n > 0$ we have

$$\left\| \sum_{k=n}^{m} s_k\mathbf{v}_k \right\|^2 = \sum_{k=n}^{m} (s_k)^2 \tag{1}$$

by the Pythagorean Formula. If $\sum_{k=1}^{\infty}(s_k)^2 < \infty$, then the sequence of partial sums $s_m = \sum_{k=1}^{m} s_k\mathbf{v}_k$ is a Cauchy sequence by (1). Therefore the series $\sum_{k=1}^{\infty} s_k\mathbf{v}_k$ converges since H is complete.

Conversely, if the series $\sum_{k=1}^{\infty} s_k\mathbf{v}_k$ converges, then (1) implies the convergence of $\sum_{k=1}^{\infty}(s_k)^2$ since the sequence of partial sums $\sigma_m = \sum_{k=1}^{m}(s_k)^2$ is a Cauchy sequence in $\mathbb{R}$, and Cauchy sequences converge in $\mathbb{R}$ (i.e. $\mathbb{R}$ is complete).

We can show

$$\left\|\sum_{k=1}^{\infty} s_k \mathbf{v}_k\right\|^2 = \sum_{k=1}^{\infty}(s_k)^2$$

by taking $n = 1$ and letting $m \to \infty$ in (1). ∎

Proof of the Fundamental Theorem. Let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots\}$ be a basis of infinite dimensional vector space H (with inner product $\langle \cdot, \cdot \rangle$) and let $\mathbf{b} \in H$. Define $T(\mathbf{b}) = \{\alpha_k\}$ (a sequence) where $\alpha_k = \langle \mathbf{b}, \mathbf{b}_k \rangle$ for $k \in \mathbb{N}$. By Lemma 2, T is a one-to-one mapping from H onto l^2. Since the inner product is linear, then T is linear. T also preserves inner products:

$$\begin{aligned}
\langle T(\mathbf{x}), T(\mathbf{y}) \rangle &= \langle \{\alpha_k\}, \{\beta_k\} \rangle \\
&= \sum_{k=1}^{\infty} \alpha_k \beta_k \\
&= \sum_{k=1}^{\infty} \langle \mathbf{x}, \mathbf{b}_n \rangle \langle \mathbf{y}, \mathbf{b}_k \rangle \\
&= \sum_{k=1}^{\infty} \langle \mathbf{x}, \langle \mathbf{y}, \mathbf{b}_k \rangle \mathbf{b}_k \rangle \\
&= \left\langle \mathbf{x}, \sum_{k=1}^{\infty} \langle \mathbf{y}, \mathbf{b}_k \rangle \mathbf{b}_k \right\rangle \\
&= \langle \mathbf{x}, \mathbf{y} \rangle
\end{aligned}$$

Therefore T is an isomorphism from inner product space H to l^2. ∎

Note. We can also completely classify (bounded) linear operators on infinite dimensional vector spaces:

Theorem. A (bounded) linear operator on an infinite dimensional vector space with basis $\{\mathbf{b}_1, \mathbf{b}_2, \dots\}$ can be represented by an infinite matrix.

The proof follows exactly the same as in the finite dimensional case.

Applications (and Motivation)

Note. Having established that all complete infinite dimensional inner product spaces are isomorphic to l^2 (and having some intuitive way to visualize l^2 by analogy with $\mathbb{R}^n$), we have prepared students for several examples that are useful in upper level classes.

Example 1. Heine-Borel Not Valid in l^2. The set of vectors that form the standard basis for l^2

$$B = \{(1,0,0,\ldots),(0,1,0,\ldots),\ldots,(0,\ldots,0,1,0,\ldots),\ldots\}$$

is a closed and bounded set, but it is not compact.

Example 2. Fourier Series. The sequence $\{\Phi_k(x)\} = \left\{\dfrac{e^{ikx}}{\sqrt{2\pi}}\right\}$ for $k \in \mathbb{Z}$ is an orthonormal basis for $L^2([-\pi,\pi])$. Hence, $f \in L^2([-\pi,\pi])$ means that $f : \mathbb{R} \to \mathbb{C}$ and $\displaystyle\int_{-\pi}^{\pi} |f(x)|^2\,dx < \infty$ and the inner product is $\langle \mathbf{f}, \mathbf{g}\rangle = \displaystyle\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$.

Example 3. Legendre Polynomials. The Legendre Polynomials

$$\begin{aligned} P_0 &= \sqrt{\frac{1}{2}} \\ P_k(x) &= \frac{\sqrt{k+\frac{1}{2}}}{2^k k!}\frac{d^k}{dx^k}[(x^2-1)^k], k \in \mathbb{Z}^+ \end{aligned}$$

form an orthonormal basis for $L^2([-1,1])$. In fact, we get the Legendre Polynomials by applying the Gram-Schmidt process to the sequence of functions $\{x^k\}$. Several other collections of special functions result in a similar way, including the Hermite Polynomials and the Laguerre Polynomials.

Example 4. Quantum Mechanics. In quantum mechanics, a system is represented by a *state vector* $\Psi(x)$. The state vector is *normalized* when $\|\Psi(x)\|^2 = 1$. An *observable* (such as position, momentum, or spin) is represented by a Hermitian operator. If $\{\Psi_k\}$ is an eigenbasis for an operator, then the state vector can be written as

$$\Psi(x) = \sum_{k=1}^{\infty} \langle \Psi, \Psi_k \rangle \Psi_k$$

and the probability that the observable takes on eigenstate Ψ_k is $|\langle \Psi, \Psi_k \rangle|^2$. Notice that (since Ψ is normalized)

$$\sum_{k=1}^{\infty} |\langle \Psi, \Psi_k \rangle|^2 = \|\Psi\|^2 = 1.$$

CONCLUSION

Note. The above approach allows us to discuss infinite dimensional vector spaces (that is, separable Hilbert spaces) and to discuss their geometry. The examples which use L^2 spaces, however, cannot be **fully** appreciated without exploring the topics of Lebesgue measure and integration. Since these spaces are isomorphic to l^2, we do gain some insight about them (especially their geometry!).