General Physics Labs I (PHYS-2011)
EXPERIMENT MEAS–3:
The Force Table: Trig Review & Vectors

1 Introduction

Physics makes use of scalar and vector quantities to describe the world around us. After we discuss the distinction between scalar and vector physical quantities, we will focus on the representation and the combination of vector quantities. This will include a brief review of the necessary trigonometry.

2 Scalars and Vectors

A **scalar** quantity has magnitude but no directional information. Meanwhile, a **vector** has both magnitude and directional information.

Examples of scalar quantities include mass m, temperature T, and the Universal Gravitational Constant G. Examples of vectors include velocity $\vec{v}$, acceleration $\vec{a}$, and force $\vec{F}$. We distinguish *vector velocity* $\vec{v}$, which is expressed with a magnitude and direction, from *scalar speed* v, which is expressed with a magnitude only. See the following list of measurements for additional examples:

- 4 kg and 600 K are scalars.
- 420 km/s is a scalar (*i.e.*, speed).
- 420 km/s to the NW (northwest) is a vector (*i.e.*, velocity).
- 420 km/s NW is not equal to 420 km/s SE (southeast)!
- Note that textbooks and lab manuals can represent a vector with either an arrow over the variable letter (*e.g.*, $\vec{A}$), or indicate a vector with a boldface letter (*e.g.*, $\boldsymbol{A}$). In these Lab Experiments for the *General Physics Laboratory* courses (PHYS-2011 and PHYS-2021), vectors will be indicated with an arrow over the variable letter.

Arithmetic for scalars and vectors are handled differently with respect to each other. In both cases, adding or subtracting such quantities only makes sense if the quantities describe the same property and are in consistent units. Adding or subtracting vectors must account properly for directions. We will describe vector arithmetic after a brief mathematical review.

3 Basic Trigonometry

3.1 Right-Angle Triangle Relationships

Consider the triangle in the following figure — here a and b are called the '**legs**' of this triangle and c is called the **hypotenuse**, which is the side opposite the right angle (indicated with a 'square box' in the figure below).

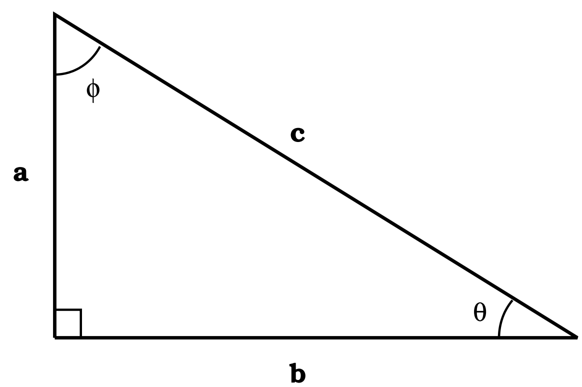

Figure 1: A right-angle triangle.

Note that the sum of the internal angles of a triangle is equal to 180°, as such, $\theta+\phi+90° = 180°$. The angles θ and ϕ in the diagram above must add up to 90° since the third angle is the 'right' angle (*i.e.*, 90°). The angles θ and ϕ are said to be *complementary angles*. Also note that besides 'degrees,' we can also measure angles in units of 'radians,' where π radians = 180°.

Below we define the 3 primary trigonometry functions, *sine, cosine*, and *tangent*, and we show the Pythagorean Theorem in terms of the sides of a right-angle triangle (*i.e.*, $a^2 + b^2 = c^2$) and the angular version of this theorem (*i.e.*, $\sin^2\theta + \cos^2\theta = 1$).

$$\begin{aligned}
\sin\theta = \frac{a}{c},\quad \cos\theta = \frac{b}{c},\quad \tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta} \\
a^2 + b^2 = c^2 \qquad \text{or} \qquad \sin^2\theta + \cos^2\theta = 1 \\
\theta + \phi \quad = 90° = \quad \frac{\pi}{2}\ \text{radians.}
\end{aligned}$$

So we see that the **sine** of an angle (*i.e.*, using angle θ in the figure above) is equal to the ratio of the length of the *opposite side* (*i.e.*, side a) of the angle to the length of the *hypotenuse* (*i.e.*, side c); the **cosine** of the angle is defined to be the ratio of the length of the *adjacent side* (*i.e.*, side b) to the length of the *hypotenuse*, and the **tangent** of the angle is the length of the *opposite side* to the length of the *adjacent side*.

Two triangles are said to be *similar* if they have the same internal angles as shown below. In the figure below, $\theta = 30^\circ$ and $\phi = 60^\circ$ in both triangles. When we have similar triangles, the ratio of corresponding sides in both triangles (as shown in the figure) are always equal, *regardless* of the lengths of their sides. This fact is often useful when dealing with vectors in physics. We will make use of this when we represent vector addition and subtraction graphically.

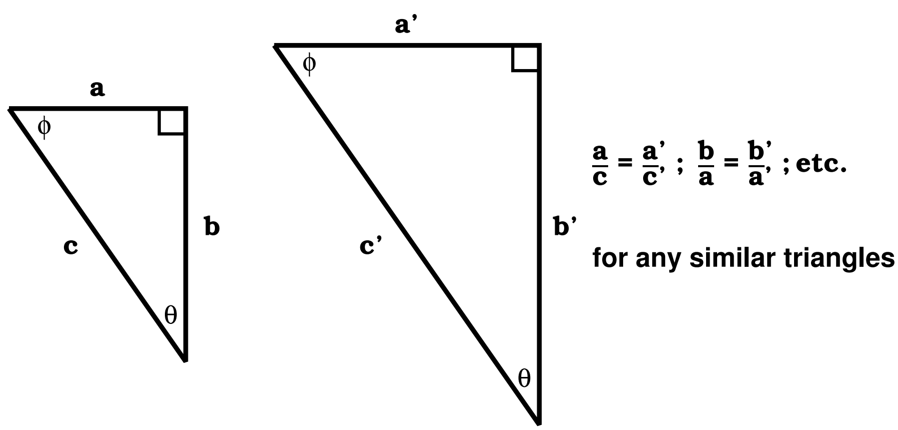

Figure 2: Similar triangles and their mathematical relationships.

All trigonometric functions have an inverse function associated with the specific function. For example, for angle θ in the triangles above we have:

$$\begin{aligned} \theta &= \sin^{-1}\left(\frac{a}{c}\right) = \cos^{-1}\left(\frac{b}{c}\right) = \tan^{-1}\left(\frac{a}{b}\right) \ , \text{ and} \\ \theta &= \sin^{-1}\left(\frac{a'}{c'}\right) = \cos^{-1}\left(\frac{b'}{c'}\right) = \tan^{-1}\left(\frac{a'}{b'}\right) \ . \end{aligned}$$

3.2 Generic Triangle Relationships

Usually, the mathematics in section 3.1 is sufficient to add and subtract vector quantities. Occasionally, the more general triangle relations summarized here may be useful:

- Law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \ .$$

- Law of cosines:

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \ , \\
b^2 &= a^2 + c^2 - 2ac\cos B \ , \\
c^2 &= a^2 + b^2 - 2ab\cos C \ .
\end{aligned}
$$

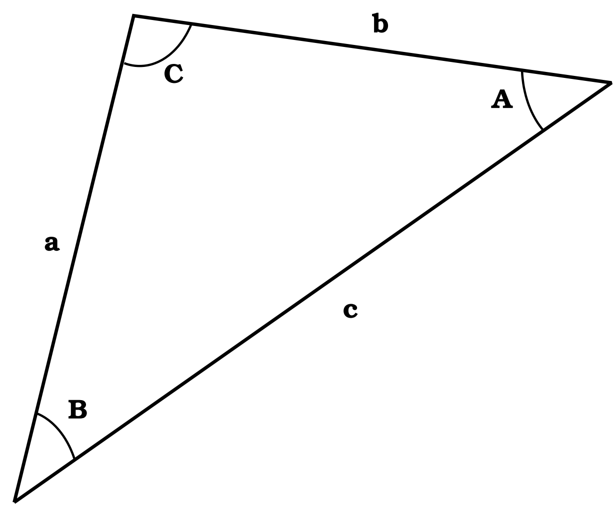

Figure 3: A generic triangle.

The equations above correspond to the generic triangle shown in Figure 3. There are other useful trigonometric identities that are sometimes useful in solving physics problems. In the equations below, angles α and β are two angles in a given generic triangle.

- Angle-sum and angle-difference relations:

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\,\cos\beta - \cos\alpha\,\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\,\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\,\tan\beta}
\end{aligned}
$$

- Double-angle relations:

$$\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\,\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \\
&= \frac{1-\tan^2\alpha}{1+\tan^2\alpha} \\
\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

4 Vectors and Vector Arithmetic

For this section, we will just focus on vectors in two dimensions, where the x-axis will correspond to the horizontal axis and the y-axis will correspond to the vertical axis. **Vectors** are represented on an x-y graph as an arrow with a distinct direction indicated by an arrowhead:

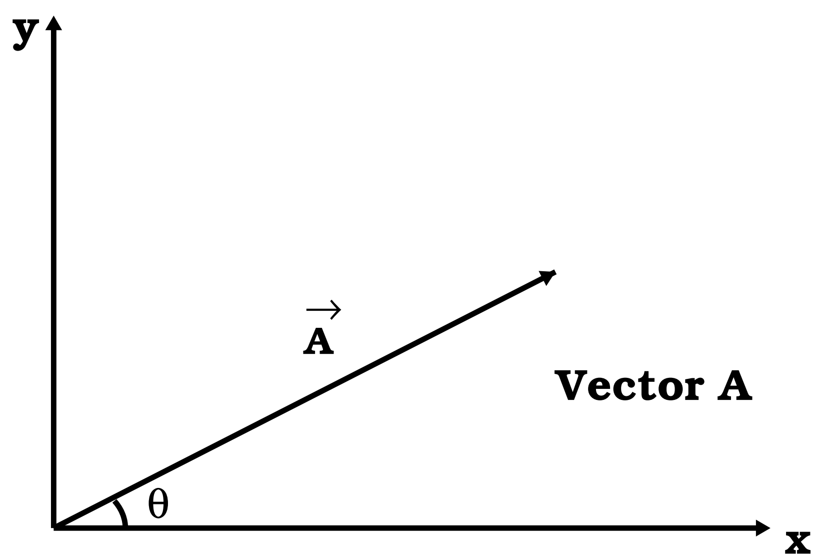

Figure 4: A graphical representation of a vector.

Note that vector $\vec{A}$ makes an angle θ with the $+x$-axis. The $+x$-axis is *usually* the reference line which the vector direction angles are measured.

4.1 Vector Addition

One can represent the addition of two or more vectors graphically by placing the *beginning* of the *2nd* vector at the *tip* of the *1st* vector. The beginning of the 3rd vector would be placed at the tip of the 2nd vector, etc. For example, let's graphically add vectors $\vec{A}$ and $\vec{B}$ in the diagrams shown in Figure 5.

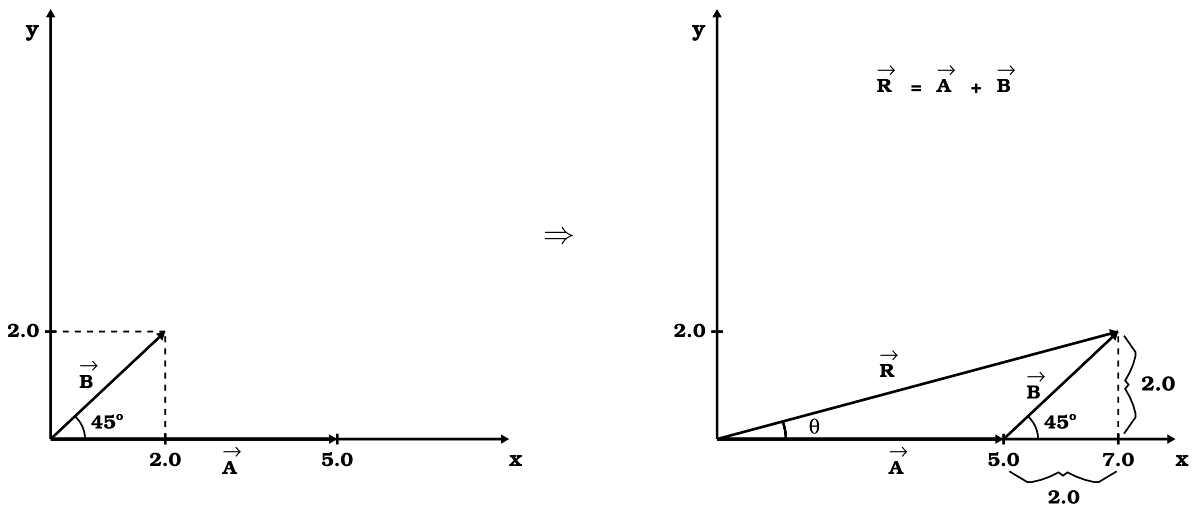

Figure 5: Adding vectors graphically.

Note how we move vector $\vec{B}$ so its tail is at the tip of vector $\vec{A}$. An arrow drawn from the tail of the first vector to the tip of the final vector represents the sum of the vectors. Our resulting vector $\vec{R} = \vec{A} + \vec{B}$ is 7 units long in the $+x$ direction and is 2 units long in the $+y$ direction. Alternatively, the vector arrows have been carefully measured with a ruler to scale to 2 and 5 units, and the angle vector $\vec{B}$ makes with the $+x$ direction was carefully measured with a protractor, one could simply measure the length of the arrow for $\vec{R}$, and the angle θ, and describe the vector sum graphically. This graphical addition will be one way we will obtain quantitative answers to today's exercises.

More often in physics, we use analytic methods. The length of the vector is determined from the Pythagorean theorem:

$$\boxed{R = \sqrt{R_x^2 + R_y^2}}$$

$\Longrightarrow$ note that the length of a vector is sometimes written as

$$\boxed{R = \left| \vec{R} \right| \ ,}$$

and that the length is always taken to be the 'positive' root of the square root. The angle that a vector makes with the x-axis is determined by any of the following equations:

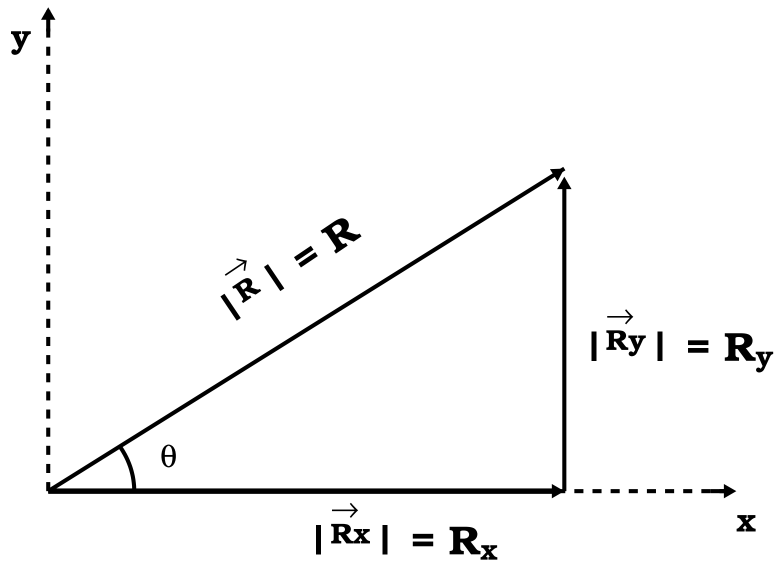

Figure 6: Components of the resultant vector.

$$\boxed{\begin{array}{rcl} R_x & = & R\cos\theta \\ R_y & = & R\sin\theta \end{array}}$$

$$\text{or} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} \Longrightarrow \quad \boxed{\tan\theta = \frac{R_y}{R_x}\ .}$$

$$\text{or} \qquad \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)\ .$$

Note that your calculator will give one mathematically valid angle. Another mathematically valid angle is 180° plus this value. It is up to you to determine which angle makes physical sense.

The basis of the analytic method for adding vectors is to express all vectors to be summed in x- and y-components. The x- and y-components are the projections of the vectors along the x- and y-axes. The x-component of the resultant vector sum is the sum of all x-components. The y-component of the resultant vector sum is the sum of all y-components.

To demonstrate this, let's ask the question, what are the lengths and angles of the 3 vectors, $\vec{A}, \vec{B}$, and $\vec{R}$ used in our graphics example of Figure 5?

$$\vec{A} \ : \ A_x = 5.0, \quad A_y = 0$$

$$
\begin{aligned}
\vec{B} &: \quad B_x = 2.0, \quad B_y = 2.0 \\
\vec{R} &: \quad R_x = A_x + B_x = 7.0, \quad R_y = A_y + B_y = 2.0
\end{aligned}
$$

$$
\begin{aligned}
A &= \sqrt{5.0^2 + 0^2} = \sqrt{25} = 5.0 \\
B &= \sqrt{2.0^2 + 2.0^2} = \sqrt{4.0 + 4.0} = \sqrt{8.0} = \sqrt{4.0 \cdot 2.0} \\
&= 2.0\sqrt{2.0} \approx 2.\underline{8}28 \approx 2.8. \\
R &= \sqrt{7.0^2 + 2.0^2} = \sqrt{49 + 4.0} = \sqrt{53} \approx 7.\underline{2}80 \approx 7.3
\end{aligned}
$$

$$
\begin{aligned}
\theta_A &= \tan^{-1}\left(\frac{0}{5.0}\right) = \tan^{-1}(0) = 0^\circ \\
\theta_B &= \tan^{-1}\left(\frac{2.0}{2.0}\right) = \tan^{-1}(1.0) = 45^\circ \\
\theta_R &= \tan^{-1}\left(\frac{2.0}{7.0}\right) \approx \tan^{-1}(0.2857) \approx 1\underline{5}^\circ.9 \approx 16^\circ \ ,
\end{aligned}
$$

Note that from this point forward, we will use the standard equals "=" sign even when the answer is not exact (as indicated above with the "≈" or '*approximately equal*' sign).

4.2 Vector Subtraction

To do vector subtraction graphically, we reverse the 2nd vector so that it is pointing in the opposite direction, then follow the rules for vector addition:

$$
\begin{aligned}
\vec{R} &= \vec{A} - \vec{B} \\
\vec{R} &= \vec{A} + (-\vec{B}).
\end{aligned}
$$

Using our graphics example for vector addition, let's subtract $\vec{B}$ from $\vec{A}$ shown graphically in Figure 7 on the next page.

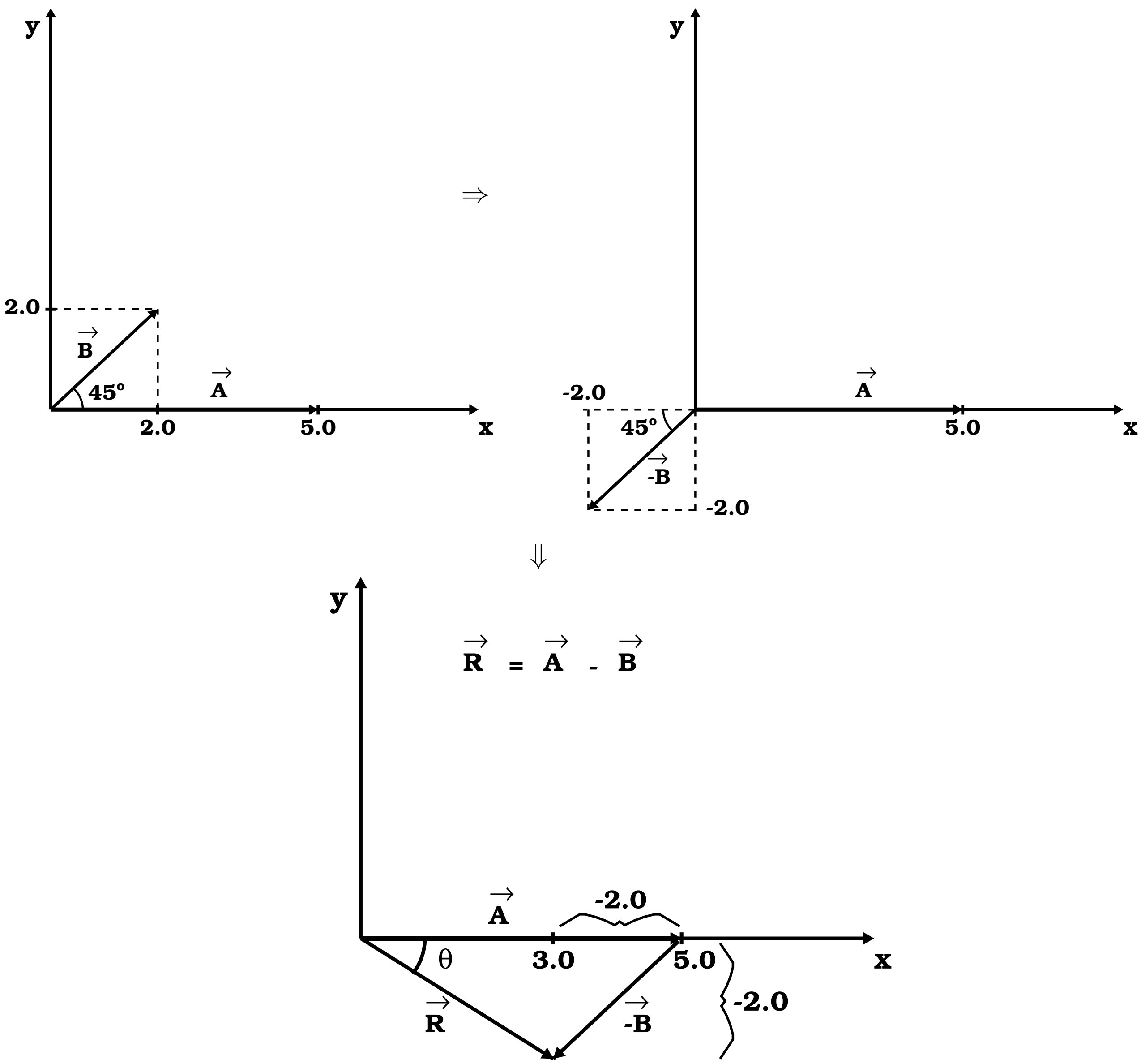

Figure 7: Subtracting vectors graphically.

therefore, $\vec{R}$ is 3 units in the x direction and is –2 units in the y direction. Careful measurement of the length of $\vec{R}$ and the new angle θ with ruler and protractor would give our graphical evaluation of $\vec{R}$.

Subtracting one vector from another algebraically is done by subtracting the respective components:

$$R_x \;=\; A_x - B_x = 5.0 - 2.0 = 3.0$$

$$R_y = A_y - B_y = 0 - 2.0 = -2.0$$

$\Longrightarrow$ *i.e.*, 3 units over in $+x$ direction, 2 units down in $-y$ direction. Now the length and direction of the resulting vector is found with:

$$\begin{aligned} R &= \left|\vec{R}\right| = \sqrt{R_x^2 + R_y^2} \\ &= \sqrt{3.0^2 + (-2.0)^2} = \sqrt{9.0 + 4.0} = \sqrt{13} = 3.\underline{6}06 = 3.6 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{-2.0}{3.0}\right) = \tan^{-1}(-0.66667) \\ &= -3\underline{3}^\circ.7 = -34^\circ \end{aligned}$$

(remember to round-off as per the significant digits of the input parameters).

5 Experimental Vector Arithmetic

A *force table* can be used to determine the sum of two or more vectors and the difference between two vectors. A picture of the force table that we use in our *General Physics I Laboratory* course is shown below in Figure 8.

Figure 8: A force table similar to the ones used in this class.

Weights (masses) will represent the magnitude of a given vector are placed on a force table in the proper direction by adjusting the angle of the individual pulleys containing the given weights — let's associate these weights with vectors $\vec{A}$ and $\vec{B}$. Then an *equilibrium* weight $\vec{E}$ is determined by trial and error by adjusting the amount of weight on the pulley and the angle placement of the pulley.

The final equilibrium weight vector will be achieved when the *central axis* of the force table is directly in the center of the *center ring* to which the pulley strings are attached. Then the **resultant vector** $\vec{R}$ of the addition of vectors $\vec{A}$ and $\vec{B}$ will be the opposite vector to vector $\vec{E}$ as shown in the following figure.

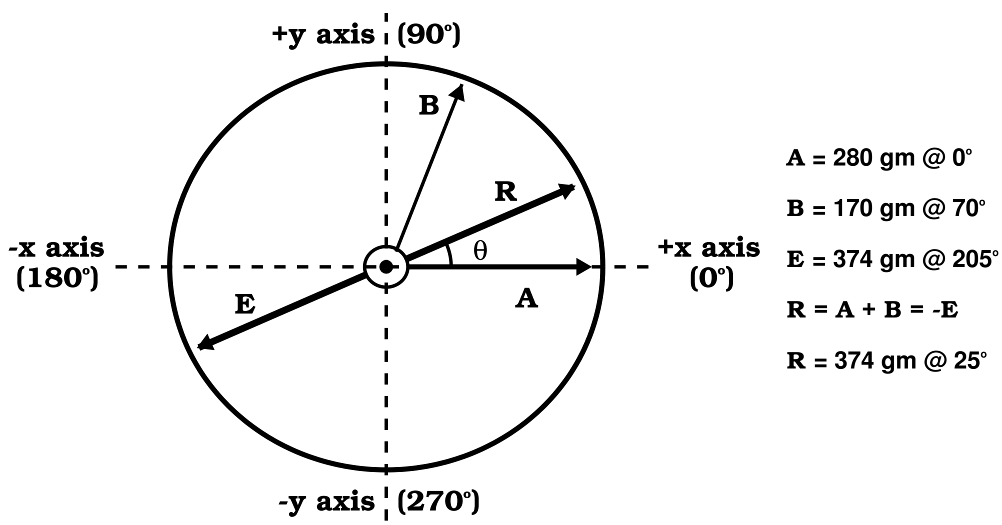

Figure 9: A graphical representation of vectors on a force table.

In this figure, the thickness of the individual vectors are indicative of the amount of weight associated with the vector. In this example, the equilibrium weight adds up to be 374 gm and the pulley is set to 205° to just balance weights associated with vectors $\vec{A}$ and $\vec{B}$. The resultant vector $\vec{R}$ then must have a magnitude of 374 gm with an angle of $\theta = 25°$, where $25° = 205° - 180°$. Note that the **mass holders** each have a mass of **50 grams**, and the masses of the holders must be included.

6 Today's Experiment Summary

On the next page, you will find instructions to carry out the exercises you must carry out for today's experiment. For the first three, you must carry out the exercise using the *graphical*, *experimental*, and *analytical* methods. For the fourth exercise, you are to use only the *graphical* and the *analytical* methods. **Make sure you use graph paper when carry out the graphical method!** You can print out such graph paper from the course web page. You will need a ruler and a protractor for a quantitative result.

Exercises

For each Exercise 1, 2, and 3, solve by *graphical*, *experimental*, and *analytical* methods. Also, compare the results obtained by the graphical and experimental methods to the analytic result, assuming the analytic result to be "correct." For Exercise 4 use the *graphical* and *analytical* methods. **Present your results in a table.**

1. Given the vectors:

$$\begin{aligned} \vec{A} &= 340 \text{ gm @ } 60^\circ, \text{ and} \\ \vec{B} &= 280 \text{ gm @ } 270^\circ, \end{aligned}$$

   find $\vec{R} = \vec{A} + \vec{B}$.

2. Given the vectors:

$$\begin{aligned} \vec{A} &= 140 \text{ gm @ } 0^\circ, \\ \vec{B} &= 300 \text{ gm @ } 80^\circ, \text{ and} \\ \vec{C} &= 250 \text{ gm @ } 240^\circ, \end{aligned}$$

   find $\vec{R} = \vec{A} + \vec{B} + \vec{C}$.

3. Given the vectors:

$$\begin{aligned} \vec{A} &= 220 \text{ gm @ } 0^\circ, \text{ and} \\ \vec{B} &= \text{unknown.} \end{aligned}$$

   The sum is known to be $\vec{R} = \vec{A} + \vec{B} = 180$ gm @ 230°. Find the unknown vector $\vec{B}$.

4. A person in a canoe can paddle his canoe at a steady 3.6 m/s in still water. He wishes to cross a 2.8 km wide river that has a current of 1.6 m/s traveling in the northward direction. Give step-by-step details in your solutions.

   (a) If this person first "aims" his canoe straight across the river in the eastern direction, the current will carry him downstream as he paddles across. What will be his actual velocity (magnitude and angle with respect to the eastward direction) as he crosses? How long will it take him to cross the river?

(b) If he "aims" the canoe somewhat upstream, he can actually travel straight across the river. In what direction must he aim? What is his actual speed across the river for this situation, and how long will it take him to cross? (*Hint:* The downstream direction will found in part (a) will <u>not</u> be the same as the upstream angle required in part (b)!) [*Hint:* You will need a **compass** for the graphical method.]