PHYS-4007/5007
COMPUTATIONAL PHYSICS PROJECT:
Polytropic Modeling of Stellar Interiors

1 Introduction

This Computer Class Project will involve working with an IDL computer code to solve a set of first-order, ordinary differential equations used to model polytropes as described below. The course project web site has been created that contains IDL procedure files to help you do this modeling, a copy of this document written in LaTeX, which you can use as a template for writing up your manuscript for this project, and the PDF output of this LaTeX document.

These Instructions contain a short description of polytropes, information on what you need to figure out in the supplied codes, and an outline on how you should format your manuscript. Those of you taking this course for honors or graduate credit will be expected to carry out additional analysis as will be described in these Instructions.

Each student will be expected to add additional lines of code to the “driving” procedure “polytrope.pro” as described in these Instructions. The “polytrope.pro” code will generate plots to the computer terminal and to “hardcopy” files (both a normal [‘.ps’] and an encapsulated [‘.eps’] **postscript** files). Remember that an encapsulated postscript plot can be imported into the LaTeX file that you will need to write for this project (see §5 of these Instructions).

2 The Physics of Polytropes

In thermal physics, matter is said to follow an **equation of state** that typically describes the internal pressure P of an object as a function of temperature T and density ρ (or volume V). Many of you are already familiar with the equation of state of an ideal gas which can be written in one of three forms:

1. The equation of state of an ideal gas as used by chemists is

$$PV = nRT \ , \tag{1}$$

where P is the gas pressure [Pa = N/m^2 = J/m], V is volume [m^3] in which the gas is contained, R is the universal gas constant (= 8.314 J/mol·K), T is the temperature [K], and n is number of 'moles' in the gas [mol]:

- Equal volume of gas at the same temperature and pressure contain the same number of molecules.
- One mole quantity of all gases at standard temperature (0°C = 273.15 K) and pressure (1 atm = 101.3 kPa) [≡ **STP**] contain the same number of molecules.

2. The equation of state of an ideal gas can also be expressed as

$$P = N k_{\mathrm{B}} T \ , \tag{2}$$

which is often used in planetary and stellar atmospheric modeling. In this equation, N is the particle density [m^{-3}] and k_{B} is Boltzmann's constant (= 1.38×10^{-23} J/K).

3. Finally, the equation of state of an ideal gas can also be written as

$$P = \frac{\rho k_{\mathrm{B}} T}{\mu m_{\mathrm{H}}} \ , \tag{3}$$

which is often used in higher-density thermal physics including internal stellar modeling. Here, ρ is the mass density [m^{-3}] and μm_{H} measures the average mass of the particles in the sample, where μ is called the mean molecular weight [unitless] and m_{H} is the mass of the hydrogen atom (= 1.6726×10^{-27} kg).

2.1 The Lane-Emden Equation

Whenever you have a *static* volume of gas, there are two competing forces that keep this gas stable: (1) The force per unit volume from the internal pressure of a thin layer of gas in the volume, $F/V = dP \cdot A/V$, and (2) the weight per unit volume of the gas in that layer trying to pull the gas to the lowest potential: $F_g/V = -mg/V = -\rho g$, where g is the acceleration due to gravity. If we have a column of gas, then $A = \pi r^2$ and $V = \pi r^2 \, dz$, where r is the radius of the column cylinder and dz is the thickness of that thin layer in the column. Equating the two forces gives

$$\frac{dP}{dz} = -\rho g \ . \tag{4}$$

This is the equation of **hydrostatic equilibrium** (HSE) for gas in a *plane-parallel* medium.

Let us now explore a spherically symmetric volume of gas, such as a star. For this geometry, HSE follows

$$\frac{dP}{dr} = -\rho \frac{G M_r}{r^2} \ , \tag{5}$$

where r is the distance along the radius of the gas sphere and M_r is the mass within the sphere measured at r. Note that in order to conserve mass,

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \; . \tag{6}$$

Around the beginning of the 20th century, several notable physicists, such as Kelvin, Ritter, Lane, Emden, and Fowler, considered the question of what limitations could be placed on the structure of a star just from the condition of HSE alone. They concluded that some other condition relating the physical parameters in the stellar interior must be used in order to specify the structure. To simplify this work, they assumed an adiabatic "quasi-static" change of state of the gas following:

$$P = K\rho^{\gamma} = K\rho^{(n+1)/n}, \tag{7}$$

where K is a constant that depends upon the nature of the polytrope, γ is called the **adiabatic index** of the gas, and n is the **polytrope index** which is also constant. This equation is called the **polytropic equation of state**. From this equation it can easily be shown that

$$n = \frac{1}{\gamma - 1} \; . \tag{8}$$

If we multiply the spherically symmetric HSE equation (*i.e.*, Eq. 5) by r^2, take the derivative of both sides with respect to r, multiply both sides by $1/r^2$, then use the mass-conservation equation (*i.e.*, Eq. 6) to eliminate dM_r/dr, we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi \, G \, \rho \; . \tag{9}$$

Now we will change variables in order to put Eq. (9) in non-dimensional form. Let's define

$$\rho = \rho_c \, \theta^{\,n} \; , \tag{10}$$

where ρ_c is the central density, θ is a dimensionless parameter, and n is the polytrope index. If we plug Eq. (10) into Eq. (9) and collect all of the numerical coefficients on the left-hand side of the equation, we get

$$\left[(n+1)\frac{K}{4\pi \, G}\, \rho^{(1-n)/n}\right] \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) = -\theta^{\,n} \; . \tag{11}$$

Because θ is dimensionless, the '[ ]' term must have units of length^2. So we will define:

$$\begin{aligned} r &= a\,\xi & (12) \\ a &= \left[(n+1)\frac{K}{4\pi \, G}\,\rho_c^{(1-n)/n}\right]^{1/2} \\ &= \left[(n+1)\frac{K}{4\pi \, G}\right]^{1/2} \rho_c^{(1-n)/2n}, & (13) \end{aligned}$$

where ξ is dimensionless. Substituting Eqs. (12) and (13) into Eq. (11) then gives

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \ .} \tag{14}$$

This is the famed **Lane-Emden Equation** of thermodynamics.

2.2 Solving the Lane-Emden Equation Using 4th-Order Runge-Kutta

Since we have a second-order differential equation, we need to do a variable substitution in order to break the Runge-Kutta Equation (Eq. 14) into two first-order differential equations. Then we can make use of the 4th-order Runge-Kutta technique to solve this set of simultaneous differential equations. If we introduce

$$\phi = \frac{d\theta}{d\xi} \ , \tag{15}$$

and insert this into Eq. (14), we can rewrite the Lane-Emden Equation as

$$\frac{d\phi}{d\xi} + \frac{2}{\xi}\,\phi = -\theta^n. \tag{16}$$

In the 4th-order Runge-Kutta scheme, we introduce a **step size** h and determine the next point $(i+1)$ from our current point (i) of our dependent variables with

$$\begin{aligned}
\phi_{i+1} &= \phi_i + \frac{1}{6}\,(k_0 + 2k_1 + 2k_2 + k_3) && (17)\\
\theta_{i+1} &= \theta_i + \frac{1}{6}\,(\ell_0 + 2\ell_1 + 2\ell_2 + \ell_3) \ , && (18)
\end{aligned}$$

where the coefficients are given by

$$\begin{aligned}
k_0 &= -h\left(2\frac{\phi_i}{\xi_i} + \theta_i^n\right) && (19)\\
\ell_0 &= h\,\phi_i && (20)\\
k_1 &= -h\left[2\left(\frac{\phi_i + k_0/2}{\xi_i + h/2}\right) + (\theta_i + \ell_0/2)^n\right] && (21)\\
\ell_1 &= h\,(\phi_i + k_0/2) && (22)\\
k_2 &= -h\left[2\left(\frac{\phi_i + k_1/2}{\xi_i + h/2}\right) + (\theta_i + \ell_1/2)^n\right] && (23)\\
\ell_2 &= h\,(\phi_i + k_1/2) && (24)\\
k_3 &= -h\left[2\left(\frac{\phi_i + k_2}{\xi_i + h}\right) + (\theta_i + \ell_2)^n\right] && (25)\\
\ell_3 &= h\,(\phi_i + k_2) \ . && (26)
\end{aligned}$$

Note that a problem arises at the first step – an indefinite value for ϕ/ξ exists. This problem can be solved by expanding θ as a function of ξ in a power series of the form

$$\theta = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \cdots \tag{27}$$

as described by Chandrasekhar (1967) and Clayton (1968). Taking derivatives with respect to ξ, dividing by ξ and taking the limit with $\xi \to 0$ gives $\phi_0/\xi_0 = -1/3$. With this, k_0 can be calculated for the first step. To insure that the integration is being carried out correctly, the values of θ from the Runge-Kutta scheme are compared to the values of θ from this expansion when $\xi < 0.1$. If the ratio of these two values are less than one-half or greater than two, the program aborts and the code reports the value of this ratio to the user. (Note that you should not encounter this issue with the values of n you are to use.)

2.3 Boundary Conditions

To get a unique solution to the Lane-Emden Equation, we need to specify two boundary conditions for this second-order differential equation. A realistic model cannot have a 'cusp' at the origin which means that

$$\begin{array}{rl} \theta = 1 & \text{at} \quad \xi = 0 \\ \frac{d\theta}{d\xi} = 0 & \text{at} \quad \xi = 0 \; . \end{array} \tag{28}$$

The Lane-Emden Equation and these boundary conditions give a unique solution for a given n written as θ_n and called the *the Lane-Emden function of index n.*

Because $\rho \sim \theta^n$, only $\theta \geq 0$ can be realistic. So the surface of the polytrope is defined as the radius at which θ first becomes zero. This is designated as ξ_s, so

$$R = a\,\xi_s \; . \tag{29}$$

This surface condition is ascertained in the code when $\theta < 10^{-10}$. If θ should become negative, either when the surface criterion is being tested or when the coefficients are being calculated, the code steps back to the last value of ϕ and θ, decreases the step size by one-half, then re-evaluates ϕ and θ.

If the surface seems to be approaching infinity in size, as will be the case when $n = 5$, the code will stop the integration. The criterion used in the code to detect this infinite surface is based upon $|d\theta/d\xi| \leq 10^{-5}$ and $\theta < 5 \times 10^{-3}$. Note that the slope criterion would not be valid if the function was to curve upward just before θ reaching zero. This is why the second criterion is added to this infinite surface criterion.

2.4 The Mass-Radius Relation

If we use Eq. (13), the above equation (*i.e.*, Eq. 29) can be written as

$$R = \left[(n+1)\,\frac{K}{4\pi\,G}\right]^{1/2} \rho_c^{(1-n)/2n}\,\xi_s\ . \tag{30}$$

Notice that $n = 1$ is a special case that divides distinctive behaviors:

$$\begin{array}{ll} n < 1 & R\uparrow\ \text{as}\ \rho_c\uparrow \\ n > 1 & R\downarrow\ \text{as}\ \rho_c\uparrow \end{array}$$

By using the mass-conservation equation (*i.e.*, Eq. 6) and the Lane-Emden Equation (*i.e.*, Eq. 14), one can show that

$$M = \int_0^R 4\pi\, r^2\,\rho\, dr = 4\pi\, a^3 \rho_c \int_0^{\xi_s} \xi^2 \theta^n\, d\xi\ ,$$

$$M = -4\pi\left[(n+1)\,\frac{K}{4\pi\,G}\right]^{3/2} \rho_c^{(3-n)/2n}\,\xi_s^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi_s}\ . \tag{31}$$

Note that the '$(\ )_{\xi_s}$' term means that this is the calculated value at the surface. Notice that $n = 3$ is a special case:

$$\begin{array}{ll} n < 3 & M\uparrow\ \text{as}\ \rho_c\uparrow \\ n > 3 & M\downarrow\ \text{as}\ \rho_c\uparrow \end{array}$$

Equations (30) and (31) can be combined by eliminating the ρ_c resulting in

$$GM^{(n-1)/n} R^{(3-n)/n} = \frac{(n+1)\,K}{(4\pi)^{1/n}} \left[-\xi^{(n+1)/(n-1)}\,\frac{d\theta_n}{d\xi}\right]_{\xi_s}^{(n-1)/n} \tag{32}$$

which produces the relation

$$\boxed{M = C(n,K)\, R^{(n-3)/(n-1)}\ ,} \tag{33}$$

where C is a constant that depends upon the polytrope index n and the pressure-density scale factor K.

3 The Course Project Web Site

I have linked the Course Project web page on the course home page — the full address of this Course Project web site is

http://www.etsu.edu/physics/lutter/courses/phys4007/project_polytrope/polytrope.htm

All of the files that are described in these Instructions can be found on this web page, including the LaTeX file that generated these Instructions.

You will need to down-load all of the IDL procedures tabulated on the Course Project web site listed in the following table. Note that you should store these files on your local machine in a subdirectory/folder specifically made for this Polytrope project – let's assume this subdirectory/folder is called "polytrope" stored in your "idl" subdirectory.

polytrope.pro:	The main driving procedure that does most of the work. You will need to add coding in this procedure where you see the '*********' comment markers.
printout.pro:	A procedure that creates, writes to the output files, and calculates surface parameters/data.
rdpolymodel.pro	A procedure to read the model data generated by polytrope.pro.
rk4coeff.pro:	Calculate the Runge-Kutta coefficients.
rk4step.pro:	Calculate the next step in the Runge-Kutta integration.

4 Description of the Project

Once these procedures have been down-loaded, students will have to add a few lines of code in **polytrope.pro** – note that the locations of these added lines, with instructions of what to do, will be indicated with

```
; *********
```

at the beginning of a line. Details of the additions you will need to code, and the analysis you will need to complete, are covered below.

4.1 Basics of the Supplied Procedure

The bulk of the coding in the polytrope.pro procedure involves defining the parameters needed to compute a polytrope model and calling the RK4COEFF and RK4STEP procedures from within a FOR/DO loop. One also could carry out the plotting requirements for this project in the latter part of this code. The procedure RK4COEFF (file rk4coeff.pro) calculates the k and ℓ coefficients of Equations (19) through (26), whereas RK4STEP (file rk4step.pro) carries out the RK4 integration (*i.e.*, Eqs. 17 and 18) and the series expansion when ξ is small (Eq. 27). The PRINTOUT (file printout.pro) will handle creating an output file (or files in our project here) containing your models data and write the data to this file. Note that you will be able to use rk4coeff.pro, rk4step.pro, rdpolymodel.pro, and printout.pro as is (but please look them over to see what they are doing). However you will need to do some coding in polytrope.pro as explained below.

4.2 Modifications to polytrope.pro

You will have to construct a FOR/DO loop within this procedure to generate your models for a set of polytrope indices. There are two sets of polytrope index data defined in this procedure: POLY_INDEX_ARRAY and POLY_INDEX_HONORS_ARRAY. The first set is to used by the regular undergraduate students and the second set is to be used by the honors undergraduate and graduate students in the class. As such, just after the NDMAX = 30000L line in the code insert lines of code similar to something like these:

```
NPOLY = N_ELEMENTS(POLY_INDEX_ARRAY)
MYOUTFILES = STRARR(NPOLY)
FOR J = 0, NPOLY-1 DO BEGIN
```

where J will be our model counter, MYOUTFILES is a string array to store the names of our output files for each polytrope model, and NPOLY is the number of polytrope indices in one of the POLY_INDEX arrays (note that this depends upon whether one is an honors/graduate student or not. Also note that NPOLY has to be defined prior to this FOR/DO statement and remember, **it has to be an integer!**

Since the other supplied procedures require a scalar value for the polytrope index, just after the FOR/DO statement, we would need a statement similar to

```
POLY_INDEX = POLY_INDEX_ARRAY[J]
```

or the honors/graduate array for those students.

The IDL coding supplied for this project will create data files for each model run. As such, we want to make sure that we have unique file names for these output file. I have already defined a ‘root’ file name (‘polyout’) and a file name ‘suffix’ (‘.txt’) in this procedure. One could make unique names by using these parameters in conjunction with our model counter (*e.g.*, ‘polyout2.txt’). To do this one could insert a new line after the POLY_INDEX line similar to

```
OUTFILE = OUTROOTNAME + STRING(J, FORMAT=’(I1)’) + OUTSUFFIXNAME
```

– note that PRINTOUT is expecting the string parameter OUTFILE as input data. Make sure you store this name in MYOUTFILES after you have created OUTFILE. In addition, at this point, you should create a specific title for the current output file making use of the default title similar to

```
TITLE = TITLEDEF + ’ # ’ + STRING(J, FORMAT=’(I1)’)
```

4.3 Plotting the Model Data

The output files that the PRINTOUT creates contains header information about the calculated model, tabular data, and a variety of Lane-Emden constants (see §4.4). The tabulated data lists the depth index [IDEPTH], ξ [XI], ϕ [PHI], θ [THETA], ξ/ξ_s [XI/XI_s], and $\rho/\overline{\rho}$ [RHO/RHOBAR]. The procedure POLYTROPE will make plots of the of the following $f(x)$ data: $\theta(\xi)$, $\theta(\xi/\xi_s)$, $\phi(\xi)$, and $\rho(\xi)/\overline{\rho}$. This procedure will also make postscript and encapsulated postscript files, where the encapsulated postscript file plot can be used in your LaTeX manuscript.

4.4 Analysis of the Polytrope Models

A few polytropic models (as defined by the polytrope index) have exact analytic solutions to certain parameters, however others require numerical solutions. Table 1 shows properties of a select set of polytropes.

Table 1: **Properties of Select Polytropes**

γ	**n**	$\boldsymbol{\xi_s}$	$\boldsymbol{\xi_s^2 \lvert d\theta/d\xi \rvert_{\xi_s}}$	$\boldsymbol{\rho_c/\overline{\rho}}$	$\mathbf{M(R)}$	**Comments**
∞	0	$\sqrt{6}$	$2\sqrt{6}$	1	$M \sim R^3$	incompressible uniform density
2	1	π	π	$\pi^2/3$	$M \sim R^\infty$	R depends only on K
5/3	3/2	3.654	2.714	5.991	$M \sim R^{-3}$	completely convective
4/3	3	6.897	2.018	54.18	$M \sim R^0$	M depends on K alone
6/5	5	∞	$\sqrt{3}$	∞	—	$n \geq 5$ polytropes are infinitely big

You will note that at the end of each output file there is a set of *Lane-Emden constants* that you will need to discuss in your paper. The first is the ratio of the central density to the average density of the polytrope which is determined algebraically from Eqs. (30) and (31):

$$\rho_c = -\frac{\xi_s}{3}\left(\frac{1}{d\theta_n/d\xi}\right)_{\xi_s}\overline{\rho} \;= -\frac{\xi_s}{3\phi_s}\,\overline{\rho}\ , \tag{34}$$

where the average density $\overline{\rho}$ is given by

$$\overline{\rho} = \frac{M}{\frac{4}{3}\pi R^3}\ . \tag{35}$$

Another parameter that is printed out near the end of the output is the **mass-function constant**, $M_{c\,\xi_s}$ which is given by

$$M_{c\,\xi_s} = -\xi_s^2\left(\frac{d\theta_n}{d\xi}\right)_{\xi_s} = -\xi_s^2\,\phi_s, \tag{36}$$

which is used in Eq. (31) to determine the mass of the polytrope. Noting Eq. (32), the output also prints the **mass-radius relation constant**:

$$\omega_n = -\xi_s^{(n+1)/(n-1)} \left(\frac{d\theta_n}{d\xi}\right)_{\xi_s} = -\xi_s^{(n+1)/(n-1)}\, \phi_s. \tag{37}$$

Note that we can rewrite Eq. (32) to solve for the polytrope equation of state constant K in Eq. (7):

$$K = N_n\, G M^{(n-1)/n} R^{(3-n)/n}, \tag{38}$$

where the K**-parameter constant**, N_n, is given by

$$N_n = \frac{1}{n+1} \left(\frac{4\pi}{\omega_n^{n-1}}\right)^{1/n} \tag{39}$$

and is also printed near the bottom of the model output file.

Realizing that at the center of our polytrope (see Eq. 28), we can calculate the central pressure of the polytrope:

$$P_c = K\, \rho_c^{(n+1)/n}. \tag{40}$$

If we substitute Eqs. (34) and (38) in Eq. (40), we obtain, after some minor transformations,

$$P_c = W_n\, \frac{GM^2}{R^4}, \tag{41}$$

where W_n represents the quantity

$$W_n = \frac{1}{4\pi\,(n+1)\,\left[(d\theta_n/d_\xi)_{\xi_s}\right]^2} = \frac{1}{4\pi\,(n+1)\,\phi_s^2}. \tag{42}$$

Note that this W_n parameter is the **pressure constant** parameter printed last in the output.

In your LaTeX manuscript, you will need to display the data for all of the models you will have to compute (as listed in the polytrope index arrays). In this analysis you should present the following data in tabular form:

n = polytrope index
$M_{c\,\xi_s}$ = mass-function constant
$\rho_c/\overline{\rho}$ = center to average density ratio
ω_n = mass-radius relation constant
N_n = K-parameter constant
W_n = pressure constant

What trends do you see in these parameters as a function of n? Do your calculated values match what is printed in Table 1? If different, to what would attribute this?

Honors and Graduate Students Only: In addition to the parameters mentioned above, you should figure out a way to label the individual curves on your plots, such as changing line styles and including a legend on the plot, or using the CURSOR command with XYOUTS to label each curve. Note that if you use the CURSOR command in your code, make sure that you only invoke it when PASS = 0, that is when plotting to the terminal. Keep in mind that XYOUTS has to be available to all devices (*e.g.*, the terminal, normal postscript, and encapsulated postscript files). Can think of a way to also display the $\rho/\overline{\rho}$ curves when $\xi < 3$ and $\rho/\overline{\rho} < 70$?

5 Writing the Final Manuscript

The Final Manuscript must be at least 6 pages in length (not including references, tables, and figures) for undergraduate students and 9 pages in length for honors and graduate students. Do not include any code listing in this manuscript! Your references, figures, and tables should take up no more than an additional 5 pages. **Note that your manuscript must be written in LaTeX!** You will need to follow the same style used in professional scientific physics and astronomy journals — break the paper up into sections:

- **Abstract**: Summary of this work and the results obtained.
- **Introduction**: Contains a discussion of what polytropes are and why they are used in thermodynamics. You are required to use at least 5 equations in this discussion.
- **Analysis**: Contains the model plots and data tables and a discussion of your analysis which involves answering the questions posed at the end of the last section (§4.4) and a description and comparison of how $\theta(\xi)$, $\theta(\xi/\xi_s)$, $\phi(\xi)$, and $\rho/\overline{\rho}$ behave on your plots for the various polytrope indices n.
- **Conclusion**: Present a detailed summary of the results of your analysis. Include some concluding remarks about this project and what you learned from it.

Feel free to use the professional journal style LaTeX template file (*i.e.*, template.tex) for your manuscript and these Instructions (file polytrope.tex) as a guide to help you write your manuscript, **however don't simply clip and paste – it needs to be written in your own words!** Email the LaTeX (.tex) file of your manuscript, the PDF file (.pdf) generated from it, and your modified polytrope.pro code to me by the project deadline date on the course syllabus. Send these files to me at both **lutter@etsu.edu** and **luttermoserd@earthlink.net**.

Acknowledgments: This project and the information presented here were inspired by Dr. Richard Durisen (1983) in his course *Stellar Interiors* in the Department of Astronomy at Indiana University.

References

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