9.1 Relations and Functions

Definitions:

- **Relation:** a set of ordered pairs.
  Example 1: {(3,1), (2,4), (5,7), (6,1), (5, 8)}
  Example 2: {(Washington, 1), (J. Adams, 2), (Jefferson, 3), ..., (Clinton, 42)}
  Example 3: {(5, a), (9, b), (4, &)}
- The **domain** of a relation is the set of all first elements in the relation. So in Example 1, the domain is {2, 3, 5, 6}. (We list each element only once.) The domain of Example 2 is {Washington, J. Adams, ..., G.H.W. Bush, Clinton}. The domain of Example 3 is {4, 5, 9}.
- The **range** of a relation is the set of all second elements of each ordered pair. So in Example 1, the range is {1, 4, 7, 8}. In Example 2, it's {1, 2, ..., 41, 42}. In Example 3, the range is {a, b, &}.
- **Function**: a relation in which no two ordered pairs have the same first element.
- Another definition of a function–a relation in which each first component corresponds to exactly one second component.

So Example 1 is *not* a function, because 5 is used as a first element in two ordered pairs. Examples 2 and 3 *are* functions, because each first element is used only once.

When we work with equations that we graph, like $3x + 4y = 12$, the set of all x-coordinates is the domain, and the set of all y-coordinates is the range.

Function Notation

We often label a function with letters of the alphabet, followed by a variable in parentheses. Common examples would be $f(x)$, $g(x)$, $h(x)$, $F(x)$, $G(x)$, $H(x)$, and $\phi(x)$. Note that $f(x)$ is a different function from $F(x)$ – in other words, case matters. The character ϕ is the Greek letter phi (in math we pronounce it "fee"). The notation $f(x)$ does NOT mean we are multiplying f and x together. It means we have a function called f that uses the variable x. When we read $f(x)$ out loud, we say "f of x."

◊Example. Given that $f(x) = x^2 - 2x + 3$, find $f(0)$.

This means replace x with 0 in the rule for $f(x)$. So to find $f(0)$, we take

$$0^2 - 2 \bullet 0 + 3 = 0 - 0 + 3 = 3.$$

So here $f(0) = 3$.

So $f(1) = 1^2 - 2 \bullet 1 + 3 = 1 - 2 + 3 = 2$.

Also, $f(-1) = (-1)^2 - 2(-1) + 3 = 1 + 2 + 3 = 6$.

$f(4) = 4^2 - 2(4) + 3 = 16 - 8 + 3 = 11$.

To determine if an equation represents a function, that is, if we can write $y = f(x)$, we solve the equation for y first. So if we were asked if the relation $\{(x,y) \mid x^2 - y^2 = 16\}$ represents a function, we would solve $x^2 - y^2 = 16$ for y and see that only one equation is possible. So let's solve that equation for y:

$-y^2 = 16 - x^2$	(subtracting x^2 from both sides)
$y^2 = x^2 - 16$	(dividing both sides by -1 and doing a little rearranging)
$y = \pm\sqrt{x^2 - 16}$	(using the Square Root Property)

Note here that y could be two different things– either a positive or negative square root. We can't have that, so this relation is not a function.

Finding the Domain of a Function

The domain of a function is the set of all numbers we can use to replace x with.

◊ That means if we had a rational function like $g(x) = \frac{4x}{3x+2}$, we have to pick all the values of x that don't cause us problems. In this case, it would mean that x couldn't be $-\frac{2}{3}$, because replacing x with $-\frac{2}{3}$ would give us a 0 in the denominator. We would say that the domain of $g(x)$ would be all numbers except -2/3. We use set notation to do this, so we would write

$$\{x \mid x \neq -\tfrac{2}{3}\}.$$

◊ Find the domain of $F(x) = \sqrt{x+3}$.

Since we only want to deal with real numbers in this chapter, the part under the radical can't be negative. So $x + 3$ has to be 0 or positive; in other words, we need $x + 3 \geq 0$. This means that $x \geq -3$. We write our domain like this:

Domain = $\{x \mid x \geq -3\}$.

We can add, subtract, multiply, and divide functions the same way we do polynomials. For example, if $G(x) = 3x + 5$ and $h(x) = 4x - 7$, then $(G + h)(x) = 7x - 2$.

The Difference Quotient

In calculus there is something used extensively called the "difference quotient." It is defined like this:

Difference Quotient = $\overline{f(a+h) - f(a)}$

We take the rule for $f(x)$, first replacing x with $(a + h)$, then replacing x with a, then subtracting the two and dividing the result by h.

♢ Find the difference quotient when $f(x) = 3x + 7$.

First, we'll find $f(a + h)$; that is, we'll replace x with $(a + h)$.

$f(a + h) = 3(a + h) + 7 = 3a + 3h + 7$

Then we find $f(a)$.

$f(a) = 3a + 7$

Next, we subtract and divide by h:

$$\frac{f(a+h) - f(a)}{h} = \frac{(3a+3h+7)-(3a+7)}{h}$$

$$= \frac{3a+3h+7-3a-7}{h} = \frac{3h}{h} = 3$$

So the difference quotient is 3.

♢ Find the difference quotient for $f(x) = 3x^2 + 9$.

Again, we'll replace x first with $a + h$ and then with a, subtract, and then divide by h. Be careful on this one! Remember to square a binomial like $a+h$, you don't simply square the two terms! Using the pattern, $(a+h)^2 = a^2 + 2ah + h^2$. So here's what our work looks like:

$$D.Q. = \frac{[3(a+h)^2 + 9] - (3a+9)}{h}$$

$$= \frac{[3(a^2 + 2ah + h^2) + 9] - 3a^2 - 9}{h}$$

$$= \frac{3a^2 + 6ah + 3h^2 + 9 - 3a^2 - 9}{h}$$

$$= \frac{6ah + 3h^2}{h}$$

$$= 6a + 3h$$

One of the many things the difference quotient is used for in calculus is finding the slope of a line that's just touching a curve at a particular point. This line is called a tangent line to the curve at that point.