CSCI 1900
Discrete Structures

Functions

Reading: Kolman, Section 5.1

Domain and Range of a Relation

The following definitions assume R is a relation from A to B.

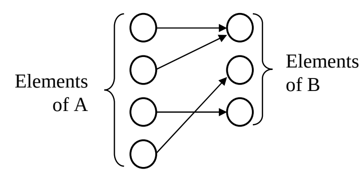
- Dom(R) = subset of A representing elements of A the make sense for R. This is called the "domain of R."
- Ran(R) = subset of B that lists all second elements of R. This is called the "range of R."

Functions

- A function *f* is a relation from A to B where each element of A that is in the domain of *f* maps to ***exactly*** one element, b, in B
- Denoted *f*(a) = b
- If an element a is not in the domain of *f*, then *f*(a) = ∅

Functions (continued)

Also called mappings or transformations because they can be viewed as rules that assign each element of A to a single element of B.



Functions (continued)

- Since *f* is a relation, then it is a subset of the Cartesian Product A × B.
- Even though there might be multiple sequence pairs that have the same element b, no two sequence pairs may have the same element a.

Functions Represented with Formulas

- It may be possible to represent a function with a formula
- Example: $f(x) = x^2$ (mapping from Z to N)
- Since function is a relation which is a subset of the Cartesian product, then it doesn't need to be represented with a formula.
- A function may just be a list of sequenced pairs

Functions not Representable with Formulas

- Example: A mapping from one finite set to another
  - A = {a, b, c, d} and B = {4, 6}
  - *f*(a) = {(a, 4), (b, 6), (c, 6), (d, 4)}
- Example: Membership functions
  - *f*(a) = {0 if a is even and 1 if a is odd}
  - A = Z and B = {0,1}

Labeled Digraphs

- A labeled digraph is a digraph in which the vertices or the edges or both are labelled with information from a set.
- A labeled digraph can be represented with functions

Examples of Labeled Digraphs

- Distances between cities:
  - vertices are cities
  - edges are distances between cities
- Organizational Charts
  - vertices are employees
- Trouble shooting flow chart
- State diagrams

Labeled Digraphs (continued)

- If V is the set of vertices and L is the set of labels of a labelled digraph, then the labelling of V can be specified by a function $f: V \rightarrow L$ where for each $v \in V$, $f(v)$ is the label we wish to attach to v.
- If E is the set of edges and L is the set of labels of a labelled digraph, then the labelling of E can be specified to be a function $g: E \rightarrow L$ where for each $e \in E$, $g(e)$ is the label we wish to attach to v.

Identity function

- The identity function is a function on A
- Denoted 1_A
- Defined by $1_A(a) = a$
- 1_A is represented as a subset of $A \times A$ with the identity matrix

Composition

- If $f: A \rightarrow B$ and $g: B \rightarrow C$, then the composition of f and g, $g \circ f$, is a relation.
- Let $a \in \text{Dom}(g \circ f)$.
  - $(g \circ f)(a) = g(f(a))$
  - If $f(a)$ maps to exactly one element, say $b \in B$, then $g(f(a)) = g(b)$
  - If $g(b)$ also maps to exactly one element, say $c \in C$, then $g(f(a)) = c$
  - Thus for each $a \in A$, $(g \circ f)(a)$ maps to exactly one element of C and $g \circ f$ is a function

Special types of functions

- *f* is "***everywhere defined***" if Dom(*f*) = A
- *f* is "***onto***" if Ran(*f*) = B
- *f* is "***one-to-one***" if it is impossible to have *f*(a) = *f*(a') if a ≠ a', i.e., if *f*(a) = *f*(a'), then a = a'
- *f*: A → B is "***invertible***" if its inverse function f^{-1} is also a function. (Note, f^{-1} is simply the reversing of the ordered pairs)

Theorems of Functions

- Let *f*: A→ B be a function; f^{-1} is a function from B to A if and only if *f* is one-to-one
- If f^{-1} is a function, then the function f^{-1} is also one-to-one
- f^{-1} is everywhere defined if and only if *f* is onto
- f^{-1} is onto if and only if *f* is everywhere defined

Another Theorem of Functions

If *f* is everywhere defined, one-to-one, and onto, then *f* is a one-to-one correspondence between A & B. Thus *f* is invertible and f^{-1} is a one-to-one correspondence between B & A.

More Theorems of Functions

- Let *f* be any function:
  - $1_B \circ f = f$
  - $f \circ 1_A = f$
- If *f* is a one-to-one correspondence between A and B, then
  - $f^{-1} \circ f = 1_A$
  - $f \circ f^{-1} = 1_B$
- Let *f*: A → B and *g*: B → C be invertible.
  - $(g \circ f)$ is invertible
  - $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$

Finite sets

- Let A and B both be finite sets with the same number of elements
- If *f*: A → B is everywhere defined, then
  - – If *f* is one-to-one, then *f* is onto
  - – If *f* is onto, then *f* is one-to-one